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$E_{7(7)}$ formulation of $N = 2$ backgrounds

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Abstract: In this paper we reformulate $N = 2$ supergravity backgrounds arising in type II string theory in terms of quantities transforming under the U-duality group $E_{7(7)}$. In particular we combine the Ramond-Ramond scalar degrees of freedom together with the $O(6, 6)$ pure spinors which govern the Neveu-Schwarz sector by considering an extended version of generalised geometry. We give $E_{7(7)}$-invariant expressions for the Kähler and hyperkähler potentials describing the moduli space of vector and hypermultiplets, demonstrating that both correspond to standard $E_{7(7)}$ coset spaces. We also find $E_{7(7)}$ expressions for the Killing prepotentials defining the scalar potential, and discuss the equations governing $N = 1$ vacua in this formalism.

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Contents

1 Introduction

Backgrounds which involve manifolds with G-structure naturally appear in string theory as generalisations of Calabi-Yau and other special holonomy compactifications [1, 2]. As for conventional special holonomy manifolds these backgrounds can be classified by the amount of supersymmetry that they leave unbroken. In both cases supersymmetry requires the existence of nowhere vanishing and globally defined spinors. This in turn reduces the structure group to a subgroup G which leaves the spinors invariant.

For special holonomy manifolds the spinors are also covariantly constant with respect to the Levi-Civita connection which is what in turn implies that the manifold has a reduced holonomy group. On the other hand, the spinors of backgrounds with G-structure are

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covariantly constant with respect to a different, torsionful connection [1, 3–5]. In type II supergravity, there are two spinors parameterising the supersymmetry. It is then natural to consider a further generalisation to $G \times G$-structures, with each spinor invariant under a different G subgroup. Geometrically this can be viewed [6] as a structure on the sum of the tangent and cotangent spaces, using the notion of "generalised geometry" first introduced by Hitchin [7, 8]. In this case one can forget the conventional geometrical structure on the manifold and discuss the background just in terms of the $G \times G$-structures. It has the advantage that these structures are often better defined globally and also can satisfy integrability conditions that are the analogues of special holonomy.

From a particle physics point of view backgrounds which leave four supercharges unbroken (corresponding to $N = 1$ in four space-time dimensions ($d = 4$)) are the most interesting. However it is often useful to first study backgrounds with additional supercharges as in this case the couplings in the effective action are more constrained. In a series of papers [9–11] we considered backgrounds with eight unbroken supercharges (corresponding to $N = 2$ in $d = 4$) and formulated them in the language of $\mathrm{SU}(3) \times \mathrm{SU}(3)$-structures.

In refs. [10, 11] we studied this problem from two different points of view. On the one hand, by losing manifest $\mathrm{SO}(9, 1)$-invariance one can rewrite the ten-dimensional supergravity in a form where only eight supercharges are manifest. This corresponds to a rewriting of the ten-dimensional action in "$N = 2$ form" though without any Kaluza-Klein reduction [12]. A slightly different point of view arises when one considers a Kaluza-Klein truncation keeping only the light modes. In this case one can integrate over the six-dimensional manifold and derive an "honest" $N = 2$ effective action in $d = 4$. In this paper we will only consider the first approach.[1]

For $G \times G$-structures the "unification" of the tangent and cotangent bundle suggests a formalism where instead of the usual tangent space structure group $GL(6, \mathbb{R})$, the group $O(6, 6)$ is used, acting on the sum of tangent and cotangent spaces. It turns out that the $N = 2$ geometry is most naturally described by two complex 32-dimensional (pure) spinors $\Phi^{\pm}$ of $O(6, 6)$ [7]. Each of them individually defines an $\mathrm{SU}(3, 3)$ structure. The magnitude and phase of $\Phi^{\pm}$ are unphysical, so each can be viewed as parameterising a point in an $O(6, 6)$ orbit corresponding to the special Kähler coset space $\mathcal{M}_{\mathrm{SK}} = O(6, 6)/\mathrm{U}(3, 3)$. The respective Kähler potentials can be expressed in terms of the square root of the $O(6, 6)$ quartic invariant built out of $\Phi^{\pm}$, known as the Hitchin function. We review these results in detail in section 2.

The $O(6, 6)$ formalism naturally captures the degrees of freedom of the NS-sector, i.e. the metric and the B-field, but it does not incorporate the Ramond-Ramond (RR) sector four-dimensional scalars into a geometrical description. One knows that for Calabi-Yau compactifications including the RR-scalars promotes the special Kähler manifold $\mathcal{M}_{\mathrm{SK}}$ into a dual quaternionic-Kähler (QK) space $\mathcal{M}_{\mathrm{QK}}$. The map $\mathcal{M}_{\mathrm{SK}} \to \mathcal{M}_{\mathrm{QK}}$ is a generic property of type II string backgrounds and is called the c-map [20, 21]. One can also consider the hyperkähler cone (or the Swann bundle) over $\mathcal{M}_{\mathrm{QK}}$ [22, 23]. Such a construction always exists and physically corresponds to the coupling of hypermultiplets to

[1] Further aspects about both effective actions are discussed, for example, in refs. [13]–[19].

superconformal supergravity [24, 25]. The hyperkähler cone has one extra quaternionic dimension corresponding to a superconformal compensator multiplet. The presence of the compensator gauges the $\mathrm{SU}(2)_{\mathrm{R}}$-symmetry of $N = 2$ together with a dilatation symmetry. The metric on the cone is then determined by a hyperkähler potential χ.

Thus the question arises if there is a generalisation of the $O(6,6)$ formalism which describes the deformation space $\mathcal{M}_{\mathrm{QK}}$. This is the topic of the present paper. By analogy with the corresponding discrete T- and U-duality groups, one wants to replace the group $O(6,6)$ of the NS-sector by $E_{7(7)}$ which acts non-trivially on all scalar fields and mixes the scalars from the NS sector with the scalars in the RR sector [26].[2] Geometrically, this "extends" Hitchin's generalised geometry and includes the RR degrees of freedom in a yet larger structure called "extended geometry" or "exceptional generalised" geometry (EGG) [27, 28]. It is important to note that $E_{7(7)}$ is not a symmetry of EGG (nor is $O(d,d)$ a symmetry of generalised geometry).[3] Instead, the construction is covariant with respect to a subgroup built from the diffeomorphism symmetry and the gauge transformations of the NS and RR form-fields, and, in addition, the objects of interest come naturally in $E_{7(7)}$ representations.

We find that the quaternionic-Kähler manifold $\mathcal{M}_{\mathrm{QK}}$ is one of the homogeneous Wolf spaces [32, 33], namely $\mathcal{M}_{\mathrm{QK}} = E_{7(7)}\,/(SO^*(12) \times \mathrm{SU}(2))$, for which the hyperkähler cone is $\mathcal{M}_{\mathrm{HKC}} = \mathbb{R}^+ \times E_{7(7)}\,/\mathrm{SO}^*(12)$. The latter space can be viewed as the moduli space of highest weight SU(2) embeddings into $E_{7(7)}$ [32]. From this construction a hyperkähler potential χ can be given in terms of the SU(2) generators [34, 35]. By decomposing $E_{7(7)}$ under its subgroup $\mathrm{SL}(2,\mathbb{R}) \times O(6,6)$ we specify explicitly the embeddings of one pure spinor, the RR potential C and the dilaton-axion. Then using the result of [35] we are able to establish agreement with the expression for χ given in [36, 37] for hyperkähler cones which generically arise via the c-map.

We also find that the space $O(6,6)/\mathrm{U}(3,3)$ can be promoted to the special $E_{7(7)}$ Kähler coset $\mathcal{M}_{\mathrm{SK}} = \mathbb{R}^+ \times E_{7(7)}\,/\,E_{6(2)}$ which again admits an action of the U-duality group $E_{7(7)}$. Furthermore its Kähler potential is given by the square root of the $E_{7(7)}$ quartic invariant built out of the **56** representation. This expression can be viewed as an $E_{7(7)}$ Hitchin function.

The $E_{7(7)}$ cosets just discussed do not appear directly in the $N = 2$ supergravity but a compatibility condition between the two spinors $\Phi^\pm$ (or more precisely, between the $SO^*(12)$ and $E_{6(2)}$ structures) has to be imposed. Furthermore, if the low-energy theory is to contain no additional massive gravitino multiplets, they either have to be integrated out or an appropriate projection is required. As these massive $N = 2$ gravitino multiplets contain scalar degrees of freedom, the scalar geometry is altered. This is reviewed in more detail in section 2.

In addition to the kinetic terms, the scalar potential can also be expressed in an $E_{7(7)}$ language, though now in a way that depends on the differential geometry of the EGG. Generically in $N = 2$ supergravity the scalar potential is given in terms of an SU(2)-triplet of Killing prepotentials $\mathcal{P}_a$. We propose an $E_{7(7)}$ form for $\mathcal{P}_a$ which coincides with

[2] In this paper we will refer to these groups loosely as T- and U-duality, though the connection to the actual discrete duality groups is only clear for toroidal compactifications.

[3] This is in contrast to more ambitious proposals such as [29]-[31].

the known expressions given in [10, 11] when decomposed under the $\mathrm{SL}(2,\mathbb{R}) \times O(6,6)$ subgroup of $E_{7(7)}$. One can also consider the form of the $N = 1$ vacuum equations in this formalism. We do not give a complete description here but show at least how the standard $O(6,6)$ equations [6] can be embedded as particular components of $E_{7(7)}$ expressions.

This paper is organised as follows. Throughout, for definiteness, we focus on the case of type IIA backgrounds, though the same formalism works equally well for type IIB. In section 2 we recall how the $N = 2$ backgrounds can be written in terms of the generalised geometrical $O(6,6)$ formalism following [10, 11]. In section 3 we include the RR degrees of freedom and formulate the combined structure in terms of an exceptional generalised geometry (EGG) [27, 28]. In particular, in section 3.1 we first give some basic $E_{7(7)}$ definitions and in 3.2 we introduce the notion of exceptional generalised geometry. Then in sections 3.3 and 3.4 we discuss the moduli spaces for the hypermultiplet and vector multiplet sectors in terms of $E_{7(7)}$ coset manifolds, specifying in particular the $\mathrm{SL}(2,\mathbb{R}) \times O(6,6)$ embedding of the NS and RR degrees of freedom. We give an $E_{7(7)}$-invariant expression for the hyperkähler potential χ following the explicit construction of ref. [35] and we also show that the Kähler potential on the vector multiplet moduli space $\mathcal{M}^{+}_{\mathrm{SK}}$ is given by the square-root of the $E_{7(7)}$ quartic invariant in complete analogy to the Hitchin function in the $O(6,6)$ case. In section 3.5 we discuss the combined vector and hypermultiplet sectors, which results in some compatibility condition between the structures in $E_{7(7)}$, as well as some constraints coming from requiring a standard supergravity action. In section 4.1 we then give an $E_{7(7)}$ expression for the Killing prepotentials. Section 4.2 discusses the $E_{7(7)}$ version of the $N = 1$ background conditions determined in ref. [6]. Section 5 contains our conclusions and some of the more technical details of the computations are presented in two appendices.

2 Review of $O(6,6)$ formalism

In this section we briefly recall some of the results of refs. [10, 11] in order to set the stage for our analysis. We will use the conventions of [11]. It is perhaps helpful to stress again that in this paper we are *not* making a dimensional reduction of type II supergravity. Rather we are rewriting the full ten-dimensional theory in a four-dimensional $N = 2$ language, where one can decompose the degrees of freedom into hypermultiplets and vector multiplets. Necessarily this requires breaking the manifest local $\mathrm{Spin}(9,1)$ symmetry to $\mathrm{Spin}(3,1) \times \mathrm{Spin}(6)$, and also that we can consistently pick out eight of the 32 supersymmetries. One can then introduce special Kähler and quaternionic moduli spaces for the corresponding scalar (with respect to $\mathrm{Spin}(3,1)$) degrees of freedom. However these degrees of freedom will still depend on the coordinates of all ten dimensions.

As an example, suppose we have a product manifold $M^{9,1} = M^{3,1} \times M^6$ with an $\mathrm{SU}(3)$ structure on M^6 defined by a two-form J and a three-form Ω, both of which are scalars with respect to $\mathrm{Spin}(3,1)$. If we had a Calabi-Yau manifold then Ω and J are constrained by requiring $\mathrm{d}J = \mathrm{d}\Omega = 0$. Let us focus on Ω. In a conventional dimensional reduction one expands Ω in terms of harmonic forms (α_A, β^A) according to $\Omega = Z^A \alpha_A - F_B(Z)\beta^B$, and then shows that there is a special Kähler moduli space $\mathcal{M}_{\mathrm{trunc}}$ for the four-dimensional fields Z^A which depends on the complex geometry of the Calabi-Yau manifold. Similarly, for manifolds of SU(3) structure, where nowhere vanishing J and Ω exist but are generically not closed, one can truncate the degrees of freedom to a finite dimensional subspace, and do a similar expansion as in Calabi-Yau manifolds, but in this case involving forms which are not necessarily harmonic. The moduli space $\mathcal{M}_{\rm trunc}$ spanned by the four-dimensional fields is still special Kähler [11, 14]. In this paper on the other hand we look at the space of all structures Ω. Rather than a finite set of moduli Z^A one can choose a different three-form Ω at each point in the six-dimensional space. The space of such Ω at a given point is $\mathbb{R}^+ \times GL(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C})$ and it turns out that this is *also* a special Kähler space $\mathcal{M}_{\rm SK}$. In summary, we have two cases, given $x \in M^{3,1}$ and $y \in M^6$

$$
\begin{aligned}
\text{untruncated:} \qquad & \Omega = \Omega(x,y) \in \Lambda^3 TM^6\,, \\
& \Omega(x,y) \in \mathcal{M}_{\rm SK} \qquad \text{at each point } (x,y) \in M^{3,1}\times M^6\ , \\
\text{finite truncation:} \qquad & \Omega = Z^A(x)\,\alpha_A(y) - F_B(Z(x))\,\beta^B(y)\,, \\
& Z^A(x) \in \mathcal{M}_{\rm trunc} \qquad \text{at each point } x \in M^{3,1}\ .
\end{aligned}
\tag{2.1}
$$

Note that $\mathcal{M}_{\rm SK} \simeq \mathbb{R}^+ \times GL(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C})$ is the same for all manifolds M^6 while $\mathcal{M}_{\rm trunc}$ depends on the particular manifold. Furthermore $\mathcal{M}_{\rm trunc}$ can be obtained from the fibration of $\mathcal{M}_{\rm SK}$ over M^6 by restricting to a finite subspace of sections Ω.

More generally in [10, 11] we simply assumed that the tangent bundle of the ten-dimensional space-time splits according to $TM^{9,1} = T^{3,1} \oplus F$, where F admits a pair of nowhere vanishing Spin(6)-spinors. Here, for simplicity, we will always consider the case where $M^{9,1} = M^{3,1} \times M^6$ so $F = TM^6$ and usually just write TM for TM^6. The split of the tangent space implies that all fields of the theory can be decomposed under $\mathrm{Spin}(3,1) \times \mathrm{Spin}(6)$. In particular one can decompose the two supersymmetry parameters of type II supergravity ϵ^1, ϵ^2 as[4]

$$
\begin{aligned}
\epsilon^1 &= \varepsilon^1_+ \otimes \eta^1_- + \varepsilon^1_- \otimes \eta^1_+\,, \\
\epsilon^2 &= \varepsilon^2_+ \otimes \eta^2_\pm + \varepsilon^2_- \otimes \eta^2_\mp\,,
\end{aligned}
\tag{2.2}
$$

where in the second line the upper sign is taken for type IIA and the lower for type IIB. Here η^I_+ with $I = 1,2$ are spinors of Spin(6) while ε^I are Weyl spinors of $\mathrm{Spin}(3,1)$.[5] We see that for a given pair (η^1_+, η^2_+) there are eight spinors parameterised by $\varepsilon^I_\pm$. These are the eight supersymmetries which remain manifest in the reformulated theory. Each of the η^I is invariant under a (different) SU(3) inside Spin(6). The two SU(3) intersect in an SU(2) and the established nomenclature calls this situation a local SU(2)-structure.

Such backgrounds have a very natural interpretation in terms of generalised geometry. Recall that this is defined in terms of the generalised tangent space

$$
E = TM \oplus T^*M \tag{2.3}
$$

[4]In section 3.5 we will find some subtleties in counting the degrees of freedom on the moduli spaces, that actually will lead us to slightly generalise this $N = 2$ spinor ansatz.

[5]In each case η^I_- and ε^I_- are the charge conjugate spinors and the $\pm$ subscripts denote the chirality (for more details see appendix A of [11]).

built from the sum of the tangent and cotangent spaces. If M is d-dimensional, there is a natural $O(d,d)$-invariant metric[6] on E, given by $\eta(Y,Y) = i_y\xi$ where $Y = y + \xi \in E$, with $y \in TM$ and $\xi \in T^*M$. One can then combine (η^1, η^2) into two 32-dimensional complex "pure" spinors $\Phi^\pm \in S^\pm$ of $O(6,6)$. They are defined as the spinor bilinears, or equivalently sums of odd or even forms,

$$\Phi^+ = \mathrm{e}^{-B}\eta^1_+\bar{\eta}^2_+ \equiv \mathrm{e}^{-B}\Phi^+_0\,, \qquad \Phi^- = \mathrm{e}^{-B}\eta^1_+\bar{\eta}^2_- \equiv \mathrm{e}^{-B}\Phi^-_0\,, \tag{2.4}$$

In the special case where the two spinors are aligned we have $\eta^1 = \eta^2 \equiv \eta$. In this case there is only a single SU(3) structure, familiar from the case of Calabi-Yau compactification, and one has

$$\Phi^+ = \mathrm{e}^{-(B+\mathrm{i}J)}\,, \qquad \Phi^- = -\mathrm{i}\mathrm{e}^{-B}\Omega\,, \tag{2.5}$$

where Ω is the complex $(3,0)$-form and J is the real $(1,1)$-form.

Each pure spinor is invariant under an SU(3,3) subgroup of $O(6,6)$ and so each individually is said to define an SU(3,3) structure on E. In particular this defines a generalised (almost) complex structure. Explicitly one can construct the invariant tensor

$$\mathcal{J}^{\pm A}{}_B \;=\; \mathrm{i}\,\frac{\langle\Phi^\pm, \Gamma^A{}_B\bar{\Phi}^\pm\rangle}{\langle\Phi^\pm, \bar{\Phi}^\pm\rangle}\,, \tag{2.6}$$

satisfying $(\mathcal{J}^\pm)^2 = -\mathbf{1}$. Here, Γ^A with $A = 1,\ldots,12$ are gamma-matrices of $O(6,6)$, Γ^{AB} are antisymmetrised products of gamma-matrices, indices are raised and lowered using η and the bracket denotes the Mukai pairing defined by

$$\langle\psi,\chi\rangle = \sum_p (-)^{[(p+1)/2]}\psi_p \wedge \chi_{6-p} \equiv (s(\psi)\wedge\chi)_6\ . \tag{2.7}$$

(The subscripts denote the degree of the component forms, and the operation s assigns the appropriate signs to the component forms. This pairing is simply the natural real bilinear on $O(6,6)$ spinors. Note that the pure spinors $\Phi^\pm$ also satisfy $\langle\Phi^+, \bar{\Phi}^+\rangle = \langle\Phi^-, \bar{\Phi}^-\rangle$.)

The generalised almost complex structures $\mathcal{J}^\pm$ also induce a decomposition of the generalised spinor bundles $S^\pm$ into modules with definite eigenvalue under the action of $\frac{1}{4}\mathcal{J}^\pm_{AB}\Gamma^{AB}$. In particular, one finds

$$\frac{1}{4}\mathcal{J}^\pm_{AB}\Gamma^{AB}\,\Phi^\pm = 3\mathrm{i}\Phi^\pm, \qquad \frac{1}{4}\mathcal{J}^\pm_{AB}\Gamma^{AB}\,\bar{\Phi}^\pm = -3\mathrm{i}\bar{\Phi}^\pm\ . \tag{2.8}$$

One can also use this action to define a coarser grading of $S^\pm$, namely an almost complex structure on $S^\pm$, first introduced in this context by Hitchin [7], and given by

$$J^\pm_{\mathrm{Hit}} = \exp\left(\frac{1}{8}\pi\mathcal{J}^\pm_{AB}\Gamma^{AB}\right)\,, \tag{2.9}$$

such that (in six-dimensions and acting on $S^\pm(E)$) one has $(J^\pm_{\mathrm{Hit}})^2 = -\mathbf{1}$ and in particular, $J^\pm_{\mathrm{Hit}}\Phi^\pm = -\mathrm{i}\Phi^\pm$.

[6] We use η to denote both the $O(d,d)$ metric and the $O(d)$ spinors η^I. The distinction between them should be clear from the context.

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One finds that the specific $\Phi^\pm$ given by (2.4) also satisfy the "compatibility" condition

$$\langle \Phi^+, \Gamma^A \Phi^- \rangle = 0 \quad \forall A \, . \tag{2.10}$$

This implies that the common stabiliser group in $O(6,6)$ of the pair (Φ^+, Φ^-) is $\mathrm{SU}(3) \times \mathrm{SU}(3)$, or equivalently that together they define an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in $O(6,6)$.

One can also view this structure in terms of the way the supergravity metric g and B-field are encoded in generalised geometry. One can combine g and B into an $O(2d)$ metric on the generalised tangent space. This is compatible with the $O(d,d)$ metric such that together they are invariant under $O(d) \times O(d)$ and hence define an $O(d) \times O(d)$ structure. Thus in the six-dimensional case one can regard g and B as parameterising the 36-dimensional (Narain) coset space $O(6,6)/O(6) \times O(6)$. The two six-dimensional spinors η^I transform separately under the two Spin(6) groups. Therefore the nowhere vanishing pair (η^1, η^2) defines a separate SU(3) structure in each $O(6)$ factor. Thus collectively we see that g, B and the pair (η^1, η^2) define an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in generalised geometry.

In summary we conclude that each pure spinor $\Phi^\pm$ defines an $\mathrm{SU}(3,3)$ structure and that each parameterises a 32-dimensional coset space [7]

$$\widetilde{\mathcal{M}}^\pm_{\mathrm{SK}} = \frac{O(6,6)}{\mathrm{SU}(3,3)} \times \mathbb{R}^+. \tag{2.11}$$

The appearance of the coset G/H can also be understood as follows. It is the orbit generated by the G-action on an element which is stabilised by H. A simple example is the sphere $S^d = \mathrm{SO}(d+1)/\mathrm{SO}(d)$, which can be seen as the orbit of the unit vector in $\mathbb{R}^{d+1}$ when acting with the group $\mathrm{SO}(d+1)$. We have precisely the same situation in that Φ^+, say, can be viewed as parameterising $O(6,6)$ orbits which are stabilised by $\mathrm{SU}(3,3)$. The $\mathbb{R}^+$ then corresponds to the freedom to additionally rescale Φ^+. In fact the real part of $\Phi^\pm$ alone is stabilised by $\mathrm{SU}(3,3)$. Since a generic real spinor is 32 dimensional, as are the orbits, we see that in this case the orbit of $\mathrm{Re}\,\Phi^\pm$ forms an open set in the space of all real spinors (a so called "stable orbit") [7].

It turns out that the magnitude and phase of $\Phi^\pm$ are not physical. Modding out by such complex rescalings gives the spaces

$$\mathcal{M}^\pm_{\mathrm{SK}} = \widetilde{\mathcal{M}}^\pm_{\mathrm{SK}}/\mathbb{C}^* \simeq \frac{O(6,6)}{\mathrm{U}(3,3)} \, . \tag{2.12}$$

As we will review below, there is a natural rigid special Kähler metric on $\widetilde{\mathcal{M}}^\pm_{\mathrm{SK}}$ and a local special Kähler metric on $\mathcal{M}^\pm_{\mathrm{SK}}$. The Kähler potentials read [7]

$$\mathrm{e}^{-K^\pm} = \mathrm{i}\langle \Phi^\pm, \bar{\Phi}^\pm \rangle \, . \tag{2.13}$$

Note that a complex rescaling of the pure spinors $\Phi^\pm$ is unphysical in that it corresponds to a Kähler transformation in $K^\pm$. This degree of freedom in $\Phi^\pm$ will be part of a superconformal compensator in the $E_{7(7)}$ formulation.

Given that the groups $\mathrm{SO}(6,6)$ and $\mathrm{SU}(3,3)$ are non-compact, the spaces $\mathcal{M}^\pm_{\mathrm{SK}}$ and $\widetilde{\mathcal{M}}^\pm_{\mathrm{SK}}$ are both non-compact and have pseudo-Riemannian metrics on them. In particular the signature of the metric on $\mathcal{M}^\pm_{\mathrm{SK}}$ is $(18, 12)$. We return to this below.

As we have mentioned above, the two $\mathrm{Re}\,\Phi^\pm$ together satisfying (2.10) define an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure inside $O(6,6)$. Therefore the compatible pair $(\mathrm{Re}\,\Phi^+,\mathrm{Re}\,\Phi^-)$ parameterises the 52-dimensional coset

$$(\mathrm{Re}\,\Phi^+,\mathrm{Re}\,\Phi^-): \qquad \widetilde{\mathcal{M}} = \frac{O(6,6)}{\mathrm{SU}(3)\times\mathrm{SU}(3)}\times\mathbb{R}^+\times\mathbb{R}^+\,. \tag{2.14}$$

(Note that the dimensionality of $\widetilde{\mathcal{M}}$ counts correctly the 2×32 degrees of freedom in $\mathrm{Re}\,\Phi^+,\mathrm{Re}\,\Phi^-$ minus the 12 compatibility constraints of (2.10).) $\widetilde{\mathcal{M}}$ is a particular slice in the product space $\mathcal{M}^+_{\mathrm{SK}}\times\mathcal{M}^-_{\mathrm{SK}}$. Again for the physical moduli space one needs to mod out by the $\mathbb{C}^*$ actions on $\Phi^\pm$, giving the 48-dimensional coset $O(6,6)/\mathrm{U}(3)\times\mathrm{U}(3)$.

Note, however, that this counting still does not match the physical NS supergravity degrees of freedom which is the 36-dimensional space of g and B, parameterising the Narain coset $O(6,6)/O(6)\times O(6)$. Furthermore, we note that the metric on $O(6,6)/\mathrm{U}(3)\times\mathrm{U}(3)$ has signature $(36,12)$. Thus there are twelve degrees of freedom in the latter coset which are not really physical (and have the wrong sign kinetic term). Under $\mathrm{SU}(3)\times\mathrm{SU}(3)$ these transform as triplets $(\mathbf{3},\mathbf{1})$, $(\mathbf{1},\mathbf{3})$ and their complex conjugates. In terms of $N=2$ supergravity, these representations are associated with the massive spin-$\frac{3}{2}$ multiplets and one expects that these directions are gauge degrees of freedom of the massive spin-$\frac{3}{2}$ multiplets. This leaves a 36-dimensional space as the physical parameter space. It would be interesting to give a geometrical interpretation of this reduction, perhaps as a symplectic reduction of $\mathcal{M}^+_{\mathrm{SK}}\times\mathcal{M}^-_{\mathrm{SK}}$ with a moment map corresponding to the constraint (2.10).

We can make this physical content explicit by using the decomposition under $\mathrm{SU}(3)\times\mathrm{SU}(3)$ to assign the deformations along the orbits of Φ^+ and Φ^- as well as the RR degrees of freedom to $N=2$ multiplets. In type IIA, the RR potential contains forms of odd degree, which from the four-dimensional point of view contribute to vectors and scalars. The vectors, having one space-time index, are even forms on the internal space and we denote them C^+_μ, while the scalars are internal odd forms denoted C^-. In order to recover the standard $N=2$ supergravity structure we imposed in refs. [10, 11] the constraint that no massive spin-$\frac{3}{2}$ multiplets appear. As we mentioned, this corresponds to projecting out any triplet of the form $(\mathbf{3},\mathbf{1}),(\mathbf{1},\mathbf{3})$ or their complex conjugates. With this projection only the gravitational multiplet together with hyper-, tensor-, and vector multiplets survive. These are shown for type IIA in table 1. (In what follows, we restrict to type IIA theory, the type IIB case follows easily by changing chiralities.) $g_{\mu\nu}$ and $C^+_{\mu\,(\mathbf{1})}$ denote the graviton and the graviphoton, respectively, which together form the bosonic components of the gravitational multiplet.[7] Φ^+ represents the scalar degrees of freedom in the vector multiplets (with the $(\mathbf{3},\bar{\mathbf{3}})$ part of C^+_μ being the vectors) while Φ^- together with C^- combines into a hypermultiplet. Finally the four-dimensional dilaton ϕ, $B_{\mu\nu}$ and the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ singlet component of $C^-_{(\mathbf{1})}$ form the universal tensor multiplet.

After requiring compatible spinors and projecting out the triplets, $N=2$ supergravity requires the scalar manifold to be

$$\mathcal{M} = \mathcal{M}^+_{\mathrm{SK}}\times\mathcal{M}_{\mathrm{QK}}\,, \tag{2.15}$$

[7]The subscript $(\mathbf{1})$ indicates that it is the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ singlet of the RR forms C^+_μ or C^-.

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multiplet	$SU(3) \times SU(3)$rep.	bosonic field content
gravity multiplet	$(\mathbf{1}, \mathbf{1})$	$g_{\mu\nu}, C^{+}_{\mu\,(\mathbf{1})}$
vector multiplets	$(\mathbf{3}, \bar{\mathbf{3}})$	C^{+}_{μ}, Φ^{+}
hypermultiplets	$(\mathbf{3}, \mathbf{3})$	Φ^{-}, C^{-}
tensor multiplet	$(\mathbf{1}, \mathbf{1})$	$B_{\mu\nu}, \phi, C^{-}_{(\mathbf{1})}$

Table 1. $N = 2$ multiplets in type IIA.

where the first factor arises from Φ^+ (or more generally from the vector multiplets), while the second factor comes from the hypermultiplets (Φ^-, C^-) and the dualised tensor multiplet. In the NS-subsector, i.e. for $C^- = 0$, one has the submanifold

$$\mathcal{M}_{\mathrm{NS}} = \mathcal{M}^{+}_{\mathrm{SK}} \times \mathcal{M}^{-}_{\mathrm{SK}} \times \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \,, \tag{2.16}$$

where the Kähler potentials of the first two factors are still given by (2.13) [9–11, 38], while for the last factor it reads

$$\mathrm{e}^{-K_S} = -\mathrm{i}(S - \bar{S}) = 2\mathrm{e}^{-2\phi} \,. \tag{2.17}$$

The four-dimensional dilaton ϕ is related to the ten-dimensional dilaton φ by $\phi = \varphi - \frac{1}{4} \ln \det g_{mn}$. Equivalently one can write

$$\mathrm{e}^{-2\phi} = \mathrm{e}^{-2\varphi}\, \mathrm{vol}_6 \,, \tag{2.18}$$

where vol_6 is the volume form on M, so $\mathrm{e}^{-2\phi}$ transforms as a six-form. In S it combines with the six-form $\tilde{B}$ corresponding to the ten-dimensional dual of $B_{\mu\nu}$, into the complex six-form field $S = \tilde{B} + \mathrm{i}\mathrm{e}^{-2\phi}$. Let us stress that, even though we are using the same notation, the individual factors in (2.16) are *not* given by (2.12). The latter only appear before applying the triplet-projection and the compatibility constraint.

In the case of a single SU(3) structure the Kähler potentials (2.13) reduce to the familiar Calabi-Yau expressions [38]. Inserting (2.5) into (2.13) one arrives at

$$\mathrm{e}^{-K^+} = \frac{4}{3} J \wedge J \wedge J \,, \qquad \mathrm{e}^{-K^-} = \mathrm{i}\Omega \wedge \bar{\Omega} \,. \tag{2.19}$$

The exponentials $\mathrm{e}^{-K^{\pm}}$ in (2.13) coincide with the Hitchin function H defined for stable spinors of $O(6,6)$. If we write $\rho^{\pm} = 2\,\mathrm{Re}\,\Phi^{\pm}$ then H is the square root of the spinor quartic invariant $q(\rho^{\pm})$ of $O(6,6)$, that is

$$\mathrm{e}^{-K^{\pm}} = H(\rho^{\pm}) = \sqrt{q(\rho^{\pm})} \,, \tag{2.20}$$

where

$$q(\rho) = \frac{1}{48} \langle \rho, \Gamma_{AB}\rho \rangle \langle \rho, \Gamma^{AB}\rho \rangle \,. \tag{2.21}$$

As was first shown by Hitchin [7], given that the Mukai pairing defines a symplectic structure, the Hitchin function encodes the complex structure (2.9) such that together they define a rigid special Kähler metric on $\widetilde{\mathcal{M}}^{\pm}_{\mathrm{SK}}$ and hence a local special Kähler metric on $\mathcal{M}^{\pm}_{\mathrm{SK}}$. In particular one can construct a second spinor $\hat{\rho}^{\pm}$ from $\partial H/\partial\rho^{\pm}$. Writing $\Phi^{\pm} = \frac{1}{2}(\rho^{\pm} + \mathrm{i}\hat{\rho}^{\pm})$, the Hitchin function is given by the expression (2.13).

To complete the description of the ten-dimensional supergravity in terms of $N = 2$ language, we must give the Killing prepotentials $\mathcal{P}_a$ which determine the $N = 2$ scalar potential. These are similarly expressed in terms of $\Phi^{\pm}$ and can be written in a $O(6,6)$ form. For type IIA they read [11][8]

$$\begin{aligned} \mathcal{P}_+ = \mathcal{P}_1 + \mathrm{i}\mathcal{P}_2 &= -2\mathrm{i}e^{\frac{1}{2}K^- + \phi}\langle \Phi^+, \mathrm{d}\Phi^- \rangle\,, \\ \mathcal{P}_3 &= -\frac{1}{\sqrt{2}}\mathrm{i}e^{2\phi}\langle \Phi^+, G \rangle\ . \end{aligned} \tag{2.22}$$

Note here we have introduced the closed RR field strengths $G = \mathrm{e}^{-B}F$ where F are the more conventional field strengths satisfying $\mathrm{d}F - H \wedge F = 0$. It will be useful to introduce a potential for G, somewhat unconventionally denoted[9] C^-,

$$G = \mathrm{e}^{-B}F \equiv \sqrt{2}\mathrm{d}C^-\,, \tag{2.23}$$

where the factor of $\sqrt{2}$ is introduced to match the $E_{7(7)}$ conventions in what follows.

3 Reformulation in terms of $E_{7(7)}$ and EGG

In this section we are extending the formalism reviewed in the previous one by including the RR degrees of freedom C^- (for definiteness, we will consider the case of type IIA). Intuitively this extension can be understood as promoting the T-duality group $O(6,6)$ to the full U-duality group $E_{7(7)}$ which acts on all degrees of freedom (not only the ones in the NS-sector) and in particular mixes NS with RR scalars. From the supergravity point of view adding RR scalars promotes the moduli space $\mathcal{M}_{\mathrm{NS}}$ given in (2.16) to the moduli space $\mathcal{M} = \mathcal{M}^{+}_{\mathrm{SK}} \times \mathcal{M}_{\mathrm{QK}}$ given in (2.15). In particular one of special Kähler manifolds ($\mathcal{M}^{-}_{\mathrm{SK}}$ for type IIA) together with the dilaton factor is enlarged to a quaternionic-Kähler component $\mathcal{M}_{\mathrm{QK}}$.

Geometrically, this formulation involves going to an extension of Hitchin's generalised geometry, called "extended" or "exceptional generalised" geometry (EGG) [27, 28]. In conventional generalised geometry the internal metric and B-field degrees of freedom are "geometrised" by considering structures on the generalised tangent space $TM \oplus T^*M$. In EGG, one further extends the tangent space, so as to completely geometrise all the degrees freedom including the RR fields C^- and the four-dimensional axion-dilaton $(\phi, \tilde{B})$, as structures on this larger "exceptional" generalised tangent space.

This section is arranged as follows. In section 3.1 we give some basic $E_{7(7)}$ definitions and in 3.2 we briefly discuss the structure of the EGG relevant to type IIA compactifications

[8] Note that here we have made an $\mathrm{SU}(2)_R$ rotation $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) \mapsto (\mathcal{P}_1, -\mathcal{P}_2, -\mathcal{P}_3)$ as compared to the expressions in [11].

[9] This potential is usually called A in the literature.

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to four dimensions. This formalism leads to the expectation that the moduli spaces $\mathcal{M}_{\rm SK}^{+}$ and $\mathcal{M}_{\rm QK}$ should be cosets of the form $E_{7(7)}/H$. This is discussed in sections 3.3 and 3.4 for the hyper- and vector multiplets respectively. In 3.3.1 we briefly review some properties of the superconformal compensator formalism which is related to the hyperkähler cone construction discussed in 3.3.2 and 3.3.3. Given the known properties of homogeneous spaces, we argue in 3.3.2 what form the coset describing the hypermultiplet moduli spaces should take. In section 3.3.3 we show it explicitly by specifying the embedding of the NS and RR degrees of freedom, and give an $E_{7(7)}$ invariant expression for the hyperkähler potential χ following the explicit construction of ref. [35], showing as well its consistency with the literature [36]. In section 3.4 we turn to the vector multiplet moduli space $\mathcal{M}_{\rm SK}^{+}$. In 3.4.1 we argue what coset it should correspond to, and in 3.4.2 we give its explicit construction. We show that there is indeed a natural special Kähler metric, generalising the construction of [7], and that the corresponding Kähler potential is given by the square-root of the $E_{7(7)}$ quartic invariant in complete analogy to the Hitchin function in the $O(6,6)$ case. In section 3.5 we discuss the hyper and vector-multiplet sectors and their compatibility.

3.1 Basic $E_{7(7)}$ group theory

The group $E_{7(7)}$ can be defined in terms of its fundamental 56-dimensional representation. It is the subgroup of $Sp(56,\mathbb{R})$ which preserves, in addition to the symplectic structure $\mathcal{S}$, a particular symmetric quartic invariant Q.

In order to make the connection to the generalised geometry formalism it will be useful to study the decomposition under

$$E_{7(7)} \supset \mathrm{SL}(2,\mathbb{R}) \times O(6,6)\,, \tag{3.1}$$

where $O(6,6)$ corresponds to T-duality symmetry of generalised geometry, while $\mathrm{SL}(2,\mathbb{R})$ is the S-duality symmetry. The latter acts on the axion-dilaton $S=\tilde{B}+\mathrm{i}\mathrm{e}^{-2\phi}$ (where $\tilde{B}$ is the six-form dual to $B_{\mu\nu}$, and $\mathrm{e}^{-2\phi}$ is the four-dimensional dilaton six-form defined in (2.18)) by fractional linear transformations.[10] The fundamental representation decomposes as

$$\begin{aligned} \mathbf{56} &= (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32})\,,\\ \lambda &= \left(\lambda^{iA},\, \lambda^{+}\right)\,, \end{aligned} \tag{3.2}$$

where $i=1,2$ labels the $\mathrm{SL}(2,\mathbb{R})$ doublet while $A=1,\dots,12$ labels the fundamental representation of $O(6,6)$. λ^{+} denotes a 32-dimensional positive-chirality $O(6,6)$ Weyl spinor.

The adjoint representation $\mathbf{133}$ decomposes as

$$\begin{aligned} \mathbf{133} &= (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32'})\,,\\ \mu &= \left(\mu^{i}{}_{j},\, \mu^{A}{}_{B},\, \mu^{i-}\right)\,. \end{aligned} \tag{3.3}$$

This choice of spinor chiralities λ^{+}, μ^{i-} is precisely the one relevant for type IIA; the corresponding expressions for type IIB would require a swap of the chiralities. The $O(6,6)$ vector

[10] This should not be confused with the S-duality in type IIB mixing the dilaton with the axion coming from the RR sector.

indices A can be raised and lowered using the $O(6,6)$ metric η_{AB}, while the $\mathrm{SL}(2,\mathbb{R})$ indices can be raised and lowered using the $\mathrm{SL}(2,\mathbb{R})$ invariant anti-symmetric tensor ϵ, so that for any given doublet v^i we define $v_i = \epsilon_{ij}v^j$ where $\epsilon_{12} = 1$ and $v^i = \epsilon^{ij}v_j$ with $\epsilon^{12} = -1$.

The $E_{7(7)}$ symplectic and quartic invariants are given by

$$\begin{aligned}
\mathcal{S}(\lambda,\lambda') &= \epsilon_{ij}\lambda^i\cdot\lambda'^j + \langle\lambda^+,\lambda'^+\rangle\,,\\
Q(\lambda) &= \frac{1}{48}\langle\lambda^+,\Gamma_{AB}\lambda^+\rangle\langle\lambda^+,\Gamma^{AB}\lambda^+\rangle\\
&\quad -\frac{1}{2}\epsilon_{ij}\lambda^i_A\lambda^j_B\langle\lambda^+,\Gamma^{AB}\lambda^+\rangle + \frac{1}{2}\epsilon_{ij}\epsilon_{kl}\left(\lambda^i\cdot\lambda^k\right)\left(\lambda^j\cdot\lambda^l\right)\,,
\end{aligned}\tag{3.4}$$

where $X\cdot Y = \eta_{AB}X^AY^B$ and Γ^{AB} is the antisymmetrised product of $O(6,6)$ gamma-matrices. The action of the adjoint representation (with parameter μ) which leaves these invariant is given by

$$\begin{aligned}
\delta\lambda^{iA} &= \mu^i{}_j\lambda^{jA} + \mu^A{}_B\lambda^{iB} + \langle\mu^{i-},\Gamma^A\lambda^+\rangle\,,\\
\delta\lambda^+ &= \frac{1}{4}\mu_{AB}\Gamma^{AB}\lambda^+ + \epsilon_{ij}\lambda^{iA}\Gamma_A\mu^{j-}\,.
\end{aligned}\tag{3.5}$$

The adjoint action on the **133** representation (with parameter μ') is given by $\delta\mu = [\mu',\mu]$ where

$$\begin{aligned}
\delta\mu^i{}_j &= \mu'^i{}_k\mu^k{}_j - \mu^i{}_k\mu'^k{}_j + \epsilon_{jk}\left(\langle\mu'^{i-},\mu^{k-}\rangle - \langle\mu^{i-},\mu'^{k-}\rangle\right)\,,\\
\delta\mu^A{}_B &= \mu'^A{}_C\mu^C{}_B - \mu^A{}_C\mu'^C{}_B + \epsilon_{ij}\langle\mu'^{i-},\Gamma^A{}_B\mu^{j-}\rangle\,,\\
\delta\mu^{i-} &= \mu'^i{}_j\mu^{j-} - \mu^i{}_j\mu'^{j-} + \frac{1}{4}\mu'_{AB}\Gamma^{AB}\mu^{i-} - \frac{1}{4}\mu_{AB}\Gamma^{AB}\mu'^{i-}\,.
\end{aligned}\tag{3.6}$$

One can also define the invariant trace in the adjoint representation

$$\mathrm{tr}\,\mu^2 = \frac{1}{2}\mu^i{}_j\mu^j{}_i + \frac{1}{4}\mu^A{}_B\mu^B{}_A + \epsilon_{ij}\langle\mu^{i-},\mu^{j-}\rangle\,.\tag{3.7}$$

Let us briefly mention a different decomposition of $E_{7(7)}$. The maximal compact subgroup of $E_{7(7)}$ is $\mathrm{SU}(8)/\mathbb{Z}_2$. In particular, in the supersymmetry transformations, the two type II spinors really transform in the fundamental representation under (the double cover) $\mathrm{SU}(8)$. The fundamental and the adjoint representation of $E_{7(7)}$ decompose under $\mathrm{SU}(8)/\mathbb{Z}_2$ as

$$\begin{aligned}
&\mathbf{56} = \mathbf{28} + \bar{\mathbf{28}}\,, && \mathbf{133} = \mathbf{63} + \mathbf{35} + \bar{\mathbf{35}}\,,\\
&\lambda = (\lambda^{\alpha\beta},\bar\lambda_{\alpha\beta})\,, && \mu = (\mu^\alpha{}_\beta,\mu^{\alpha\beta\gamma\delta},\bar\mu_{\alpha\beta\gamma\delta})\,,
\end{aligned}\tag{3.8}$$

where $\alpha = 1,\ldots,8$ denotes the fundamental of $\mathrm{SU}(8)$ and where $\lambda^{\alpha\beta}$ and $\mu^{\alpha\beta\gamma\delta}$ are totally antisymmetric, $\mu^\alpha{}_\alpha = 0$ and $(*\bar\mu)_{\alpha\beta\gamma\delta} = \mu_{\alpha\beta\gamma\delta}$ (indices are raised and lowered with the $\mathrm{SU}(8)$ Hermitian metric, constructed from the spinor conjugation matrix). The action of the adjoint representation on the fundamental representation is given by

$$\begin{aligned}
\delta\lambda^{\alpha\beta} &= \mu^\alpha{}_\gamma\lambda^{\gamma\beta} + \mu^\beta{}_\gamma\lambda^{\alpha\gamma} + \mu^{\alpha\beta\gamma\delta}\bar\lambda_{\gamma\delta},\\
\delta\bar\lambda_{\alpha\beta} &= -\mu^\gamma{}_\alpha\bar\lambda_{\gamma\beta} - \mu^\gamma{}_\beta\bar\lambda_{\alpha\gamma} + \bar\mu_{\alpha\beta\gamma\delta}\lambda^{\gamma\delta},
\end{aligned}\tag{3.9}$$

Although we will not give it here, one can use six-dimensional gamma matrices to give explicit relations between the $\mathrm{SU}(8)/\mathbb{Z}_2$ and $\mathrm{SL}(2,\mathbb{R})\times O(6,6)$ decompositions.

3.2 EGG for type IIA with $E_{7(7)}$

To define exceptional generalised geometry for type IIA compactified to four dimensions, one starts with an extended generalised tangent space (EGT) of the form[11]

$$E = TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \left(T^*M \otimes \Lambda^6 T^*M\right) \oplus \Lambda^{\text{even}} T^*M \ . \tag{3.10}$$

The first two terms correspond to the generalised tangent bundle of conventional generalised geometry and are loosely associated to the momentum and winding of string states. The next two terms can be thought of corresponding to NS five-brane and Kaluza-Klein monopole charges. The final term is isomorphic to S^+, the positive helicity Spin(6, 6) spinor bundle, and is associated to D-brane charges. The EGT space is 56-dimensional and, just as there was a natural $O(d,d)$-invariant metric on $TM \oplus T^*M$, there is a natural symplectic form $\mathcal{S}$ and symmetric quartic invariant Q on E. The group that preserves both $\mathcal{S}$ and Q is $E_{7(7)}$. Thus the analogue of the $O(d,d)$ action is a natural $E_{7(7)}$ action on E. Essentially (3.10) corresponds to a decomposition of the **56** fundamental representation of $E_{7(7)}$ under a particular $GL(6,\mathbb{R}) \subset E_{7(7)}$ which is identified with diffeomorphisms of M. The embedding of $GL(6,\mathbb{R})$ in $E_{7(7)}$ is described explicitly in appendix A. More precisely, as discussed there, E corresponds to the decomposition of the fundamental representation weighted by $(\Lambda^6 T^*M)^{1/2}$.

One can similarly decompose the adjoint **133** representation of $E_{7(7)}$ under this $GL(6,\mathbb{R})$ subgroup and finds (see (A.8))

$$\begin{aligned} A_0 = {} & (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M \\ & \oplus \mathbb{R} \oplus \Lambda^6 T^*M \oplus \Lambda^6 TM \oplus \Lambda^{\text{odd}} T^*M \oplus \Lambda^{\text{odd}} TM \ . \end{aligned} \tag{3.11}$$

The first term corresponds to the $GL(6,\mathbb{R})$ action. However, there is a second important subgroup one can obtain from taking only the p-form elements of A_0

$$B + \tilde{B} + C^- \in \Lambda^2 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^{\text{odd}} T^*M \,, \tag{3.12}$$

giving a nilpotent sub-algebra (A.11). These are the EGG analogues of the "B-shift" symmetries of generalised geometry and are in one-to-one correspondence with the form-fields of the IIA supergravity. In particular, B is the internal B-field, C^- the RR-form potentials and $\tilde{B}$ is an internal six-form corresponding to the ten-dimensional dual of $B_{\mu\nu}$. To identify B, $\tilde{B}$ and C^- in the $\mathrm{SL}(2,\mathbb{R}) \times O(6,6)$ decomposition of the adjoint representation (3.3), we note, as explained in detail in appendix A, that the embedding of $GL(6,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{R}) \times O(6,6) \subset E_{7(7)}$ breaks the $SL(2,\mathbb{R})$ symmetry, picking out a $\mathrm{SL}(2,\mathbb{R})$ vector v^i. Using this vector, we identify B, $\tilde{B}$ and C^- as the following elements in the **133**

$$\begin{aligned} \mu^i{}_j &= \tilde{B}_{1\ldots 6} v^i v_j \,, && \tilde{B} \in \Lambda^6 T^*M \,, \\ \mu^A{}_B &= \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} , && B \in \Lambda^2 T^*M \,, \\ \mu^{i-} &= v^i C^- \,, && C^- \in \Lambda^{\text{odd}} T^*M \ . \end{aligned} \tag{3.13}$$

[11]This structure was first discussed in [27] and, in an M-theory context, in [28]. For a more complete description of the geometry in this particular case see [39].

Geometrically, these form-field potentials together with the metric and dilaton encode an $SU(8)/\mathbb{Z}_2$ structure on E [28], i.e. they parameterise the coset $E_{7(7)}/(SU(8)/\mathbb{Z}_2)$. Formally this structure is an element $I \in E_{7(7)}$ that, like a complex structure, satisfies $I^2 = -\mathbf{1}$. This then defines a (exceptional generalised) metric (EGM) on E given by $G(\lambda, \lambda) = \mathcal{S}(\lambda, I\lambda)$ where $\lambda \in E$. This is the analogue of the generalised metric on $TM \oplus T^*M$. One can show that a generic EGM can be written as[12]

$$G(\lambda, \lambda) = G_0(\mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}\lambda, \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}\lambda)\,, \tag{3.14}$$

where G_0 is a specific EGM built from g and the dilaton ϕ, the form of which will not be important, and $\mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}$ are the exponentiated actions of the adjoint elements given in (3.12). Hence C^-, $\tilde{B}$, B g and ϕ encode a generic EGM, or equivalently a point in the coset $E_{7(7)}/(SL(2,\mathbb{R}) \times O(6,6))$.

If the form field strengths are nontrivial, the potentials B, $\tilde{B}$ and C^- can only be defined locally. The EGM is then really a metric on a twisted version of (3.10), where we introduce on each patch $U_{(\alpha)}$

$$\lambda_{(\alpha)} = \mathrm{e}^{C^-_{(\alpha)}}\mathrm{e}^{\tilde{B}_{(\alpha)}}\mathrm{e}^{-B_{(\alpha)}}\lambda, \tag{3.15}$$

such that on $U_{(\alpha)} \cap U_{(\beta)}$ we have a patching by gauge transformations

$$\lambda_{(\alpha)} = \mathrm{e}^{\mathrm{d}\Lambda^+_{(\alpha\beta)}}\mathrm{e}^{\mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}}\mathrm{e}^{-\mathrm{d}\Lambda_{(\alpha\beta)}}\lambda_{(\beta)}\,, \tag{3.16}$$

which implies

$$\begin{aligned}
B_{(\alpha)} &= B_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)}\,, \\
C^-_{(\alpha)} &= C^-_{(\beta)} + \mathrm{d}\Lambda^+_{(\alpha\beta)} + \mathrm{e}^{-\mathrm{d}\Lambda_{(\alpha\beta)}}C^-_{(\beta)}\,, \\
\tilde{B}_{(\alpha)} &= \tilde{B}_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} + \left\langle \mathrm{d}\Lambda^+_{(\alpha\beta)}, \mathrm{e}^{-\mathrm{d}\Lambda_{(\alpha\beta)}}C^-_{(\beta)}\right\rangle\,.
\end{aligned} \tag{3.17}$$

These correspond precisely to the gauge transformations of the relevant supergravity potentials. Comparing with (2.23), we see, in particular, that the field strengths $H = \mathrm{d}B$ and $F = \sqrt{2}\mathrm{e}^B\mathrm{d}C^-$ are gauge invariant. The transformation of $\tilde{B}$ similarly matches the form given in [40], specialised to six dimensions.

Having summarised the key components of the EGG, let us now turn to the issue of how this structure can be used to describe the hypermultiplet and vector multiplet sectors.

3.3 Hypermultiplet sector

$N = 2$ supergravity constrains the scalar degrees of freedom in the hypermultiplets to span a quaternionic-Kähler manifold $\mathcal{M}_{QK}$. Over any such manifold one can construct a hyperkähler cone $\mathcal{M}_{HKC}$ which has one additional quaternionic dimension [23, 24]. In the following section we briefly review the appearance of Kähler cones in superconformal supergravity. The metric on the cone is characterised by a single function χ known as the hyperkähler potential. In section 3.3.3 we then identify how the NS and RR degrees of freedom can be embedded into an $E_{7(7)}$ EGG structure. We show that they parameterise a coset known as a "Wolf space" [32, 33], which admits a standard construction of a hyperkähler cone [35], with an $E_{7(7)}$ invariant expression for the hyperkähler potential.

[12]The choice of sign for B is conventional, to match the usual generalised geometry B-shift.

3.3.1 Hyperkähler cones and superconformal supergravity

Superconformal supergravity has as the space-time symmetry group the superconformal group instead of the super-Poincare group. Using a compensator formalism one can construct superconformally invariant actions and then obtain Poincare supergravity as an appropriately gauge fixed version. We cannot review the entire subject here but let us recall the properties relevant for our subsequent discussion following refs. [24, 25].

In the case of $N = 2$ one adds a compensating vector multiplet and a compensating hypermultiplet to the spectrum and couples all multiplets to the Weyl supermultiplet which contains the gravitational degrees of freedom. One of the resulting features is that the $N = 2$ R-symmetry $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$ together with the dilation symmetry are gauged. Furthermore, the dimension of the scalar manifolds are enlarged by one 'unit' and its geometry is altered. For the hypermultiplets this precisely corresponds to the hyperkähler cone construction where the four additional scalar fields of the compensator can be viewed as forming a cone (with one radial direction and an S^3) over the quaternionic-Kähler base $\mathcal{M}_\mathrm{QK}$. The geometry of this cone is no longer quaternionic-Kähler but instead hyperkähler in that the three local almost complex structures of $\mathcal{M}_\mathrm{QK}$ lift to globally defined integrable structures on the cone. Conversely, a quaternionic-Kähler manifold can be viewed as a quotient

$$\mathcal{M}_\mathrm{QK} \;=\; \frac{\mathcal{M}_\mathrm{HKC}}{\mathrm{SU}(2)_\mathrm{R} \times \mathbb{R}^+} \,, \tag{3.18}$$

where $\mathbb{R}^+$ corresponds to the dilatations and the $\mathrm{SU}(2)_\mathrm{R}$ rotates the three almost complex structures of $\mathcal{M}_\mathrm{QK}$.

$\mathcal{M}_\mathrm{HKC}$ can be characterised by a hyperkähler potential χ which is simultaneously a Kähler potential for all three complex structures. A generic expression for χ in terms of the three complex structures was given in [23, 24], while for the specific case of hyperkähler cones which arise from a special geometry via the c-map, χ was determined in refs. [36, 37]. In this case a particularly simple expression results in a gauge where the $\mathrm{SU}(2)_\mathrm{R}$ is partially fixed to a $\mathrm{U}(1)_\mathrm{R}$ subgroup [36] and one finds[13]

$$\chi \;=\; G_0^{-1}\, \mathrm{e}^{-K_\mathrm{SK}} \,, \tag{3.19}$$

where K_SK is the Kähler potential of special Kähler subspace $\mathcal{M}_\mathrm{SK}$ and G_0 contains the dilaton ϕ together with the compensator corresponding to the cone direction. More precisely,[14] here $K_\mathrm{SK} = K^-$ and

$$G_0^{-1} = \frac{1}{2}\, \mathrm{e}^{\frac{1}{2}K^- - \phi} \,, \tag{3.20}$$

where in this parameterisation the compensator for the dilatations and the $\mathrm{U}(1)_\mathrm{R}$ are identified with the degrees of freedom in the pure spinor Φ^- which correspond to the complex rescaling $\Phi^- \to c\Phi^-$.[15] Inserting (2.13) one finds

$$\chi \;=\; \frac{1}{2}\mathrm{e}^{-\phi}\, \sqrt{\mathrm{i}\langle \Phi^-, \bar{\Phi}^- \rangle} \,. \tag{3.21}$$

[13]The expression for χ in an arbitrary gauge is given in [37].

[14]Note that compared to ref. [36] we have a different convention of the dilaton. The dilaton used in that paper is obtained from the dilaton here by the replacement $2\phi \to -\phi$.

[15]Note that this expression for the compensator also appears in the $N = 1$ orientifolded version analysed in ref. [41].

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On the HKC also the expression for the Killing prepotentials $\mathcal{P}_a$ change. In ref. [24] it was found

$$\mathcal{P}_{\pm}^{\rm HCK} \;=\; \chi\, {\rm e}^{\pm {\rm i}\alpha}\, \mathcal{P}_{\pm}\,, \qquad \mathcal{P}_{3}^{\rm HCK} \;=\; \chi\, \mathcal{P}_{3}\,, \tag{3.22}$$

where $\mathrm{e}^{\mathrm{i}\alpha}$ parameterises the angle variable of the $\mathrm{U}(1)_{\mathrm{R}}$, and $\mathcal{P}_a$ are the Killing prepotentials on the quaternionic space. In the notation used above $\mathrm{e}^{\mathrm{i}\alpha}$ is the phase of the scale parameter c.

3.3.2 Expected coset for hypermultiplet sector

As argued before, we expect the moduli space $\mathcal{M}_{\rm QK}$ to be a coset of the form $E_{7(7)}/H$, corresponding to defining a particular structure H on the exceptional generalised tangent space E. Given the coset structure of $\mathcal{M}^{-}_{\rm SK}$ displayed in (2.12), and the fact that this moduli space is related to $\mathcal{M}_{\rm QK}$ by the c-map $\mathcal{M}^{-}_{\rm SK} \to \mathcal{M}_{\rm QK}$, we can actually make a simple conjecture for the form of $\mathcal{M}_{\rm QK}$. For the case of special Kähler coset spaces the corresponding quaternionic spaces are known [20, 42, 43]. For our particular case, we learn that the c-map relates

$$\text{c-map}: \qquad \mathcal{M}^{-}_{\rm SK} = \frac{O(6,6)}{\mathrm{U}(3,3)} \;\to\; \mathcal{M}_{\rm QK} = \frac{E_{7(7)}}{\mathrm{SO}^*(12)\times \mathrm{SU}(2)}\,. \tag{3.23}$$

The map is usually given for the compact real versions of these groups so we have actually generalised slightly to consider particular non-compact forms[16] giving a pseudo-Riemannian metric of signature $(40,24)$. The compact version of $\mathcal{M}_{\rm QK}$ is one of the well-known Wolf spaces [32, 33] and has $\dim(\mathcal{M}_{\rm QK}) = 64$. We see that as anticipated the U-duality group $E_{7(7)}$ determines the geometry of the quaternionic-Kähler space, and corresponds to the space of $\mathrm{SO}^*(12)\times\mathrm{SU}(2)$ structures on E. Furthermore, the dimension 64 precisely matches the expected supergravity hypermultiplet degrees of freedom: 30 in Φ^+ (since it is defined modulo complex rescalings), 32 in C^+ and two more in ϕ and $\tilde{B}$. The hyperkähler cone corresponding to the Wolf space given in (3.23) is the space

$$\mathcal{M}_{\rm HKC} \;=\; \frac{E_{7(7)}}{\mathrm{SO}^*(12)} \times \mathbb{R}^+\,, \tag{3.24}$$

with $\dim(\mathcal{M}_{\rm HCK}) = 68$. This space has been studied very explicitly in the mathematical literature and in particular a hyperkähler potential χ has been given [35]. Let us now use this construction to verify our expectations.

3.3.3 Hyperkähler cone construction à la Swann

In this section we show explicitly how Φ^-, C^- together with the dilaton/axion pair $(\phi,\tilde{B})$ parameterise the 64-dimensional Wolf space $\mathcal{M}_{\rm QK} = E_{7(7)}/(\mathrm{SO}^*(12)\times\mathrm{SU}(2))$, give the construction of the corresponding hyperkähler cone metric on $\mathcal{M}_{\rm HKC}$ following [35] and derive the form of the hyperkähler potential χ.

[16] The non-compact group $\mathrm{SO}^*(2n)$ is a real form of the complex group $\mathrm{SO}(2n,\mathbb{C})$ where the elements of the corresponding Lie algebra are complex matrices of the form $\left(\begin{smallmatrix} A & B \\ -B^* & A^* \end{smallmatrix}\right)$ with $A = -A^T$ and $B = B^\dagger$. For more details see for instance [44].

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The analysis of compact symmetric spaces G/H with quaternionic geometry is due to Wolf [32] and Alekseevskii [33] who showed there is one such space for each compact simple Lie group. Swann [23] subsequently identified the corresponding hyperkähler cone structures, viewing $\mathcal{M}_{\mathrm{HKC}}$ as an orbit in the adjoint representation under the complexified G. Koback and Swann then gave an explicit expression of the hyperkähler cone [35]. Here, we will follow these constructions to give an explicit form of the quaternionic geometry on $E_{7(7)}/(\mathrm{SO}^*(12)\times\mathrm{SU}(2))$ in terms of the supergravity degrees of freedom.

The hyperkähler cone $\mathcal{M}_{\mathrm{HKC}}$ can be viewed as an orbit in the $\mathbf{133}$ adjoint representation in two ways. In the complexified version one starts with an element $K_+\in\mathfrak{e}_7^{\mathbb{C}}$ corresponding to a highest weight root in the Lie algebra. The space $\mathcal{M}_{\mathrm{HKC}}$ is then the orbit of K_+ under $E_7^{\mathbb{C}}$. In this picture K_+ is stabilised under 99 elements of $E_7^{\mathbb{C}}$ so that $\mathcal{M}_{\mathrm{HKC}}$ is a $133-99=34$ complex-dimensional space. Given a real structure, which for us means the non-compact real form $E_{7(7)}$, one can identify the complex conjugates of elements of $\mathbf{133}$. This defines $K_-=\bar{K}_+$ and hence, for each K_+ in the orbit, a particular $\mathfrak{su}(2)$ subalgebra in the real algebra $\mathfrak{e}_{7(7)}$ generated by $K_\pm$ and the corresponding $K_3\sim[K_+,K_-]$. In this second picture $\mathcal{M}_{\mathrm{HKC}}$ is the orbit of this $\mathfrak{su}(2)$ algebra, under the real group $E_{7(7)}$, together with an overall scaling, where the triplet K_a is stabilised by a 66-dimensional $\mathrm{SO}^*(12)$ subgroup of $E_{7(7)}$. The overall scaling of K_a represents the radial direction of the hyperkähler cone, while the SU(2) action on the cone is realized by the action of the $\mathfrak{su}(2)$ algebra on itself, rotating the triplet K_a.

As discussed in section 3.1, under $\mathrm{SL}(2,\mathbb{R})\times O(6,6)$ the adjoint representation of $E_{7(7)}$ decomposes as $\mu=\left(\mu^i{}_j,\,\mu^A{}_B,\,\mu^{i-}\right)$ corresponding to $\mathbf{133}=(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{66})+(\mathbf{2},\mathbf{32'})$ (see (3.3)). Given an $\mathrm{SU}(3,3)$ structure Φ_0^- as defined in (2.4), we can then identify a triplet of elements, where $K_\pm^{(0)}=K_1^{(0)}\pm\mathrm{i}K_2^{(0)}$,

$$\begin{aligned} K_+^{(0)} &= \left(0,0,u^i\Phi_0^-\right)\,, \\ K_-^{(0)} &= \left(0,0,\bar{u}^i\bar{\Phi}_0^-\right)\,, \\ K_3^{(0)} &= \frac{1}{4}\mathrm{i}\kappa^{-1}\left\langle\Phi^-,\bar{\Phi}^-\right\rangle\Big((u^i\bar{u}_j+\bar{u}^iu_j),(\mathrm{i}u\bar{u})\mathcal{J}_0^{-A}{}_B,0\Big)\,. \end{aligned} \tag{3.25}$$

We have also used the fact that $\left\langle\Phi_0^-,\bar{\Phi}_0^-\right\rangle=\left\langle\Phi^-,\bar{\Phi}^-\right\rangle$ and defined

$$\kappa=\sqrt{\frac{1}{2}\mathrm{i}\left\langle\Phi^-,\bar{\Phi}^-\right\rangle(-\mathrm{i}u\bar{u})}\,. \tag{3.26}$$

Here u^i is a complex vector transforming as a doublet under $\mathrm{SL}(2,\mathbb{R})$, $\mathcal{J}_0^-$ is the generalised complex structure (2.6) defined by Φ_0^- and we abbreviate $uv=\epsilon_{ij}u^jv^i=u_iv^i=-u^iv_i$. The triplet $(K_1^{(0)},K_2^{(0)},K_3^{(0)})$ then satisfies the (real) $\mathfrak{su}(2)$ algebra

$$\left[K_a^{(0)},K_b^{(0)}\right]=2\kappa\,\epsilon_{abc}K_c^{(0)}\,. \tag{3.27}$$

We have included an overall scaling κ in the $\mathfrak{su}(2)$ algebra since, as mentioned above, this corresponds to the radial direction on the hyperkähler cone.

We would now like to see the action of $E_{7(7)}$ on the triplet $K_a^{(0)}$ to find the dimension of the corresponding orbit. In particular we should find that the triplet is stabilised by a

66-dimensional subgroup of $E_{7(7)}$. We first note that, by definition, Φ_0^- and hence $\mathcal{J}_0^-$ are invariant under $\mathrm{SU}(3,3) \in O(6,6)$ which correspond to 35 stabilising elements. There are no elements of $\mathfrak{sl}(2,\mathbb{R})$ which leave u^i invariant, though the element

$$\left(u^i\bar{u}_j + \bar{u}^i u_j, -\frac{1}{3}(\mathrm{i}u\bar{u})\mathcal{J}_0^{-A}{}_B, 0\right) \tag{3.28}$$

in $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{so}(6,6)$ does commute with all three $K_a^{(0)}$, and also with the SU(3,3) action. Finally we have the action of elements of the form $(0,0,\mu^{i-})$. Without loss of generality we can write $\mu^{i-} = u^i\mu^- + \bar{u}^i\bar{\mu}^-$, and then find using (3.6) that, to be a stabiliser, μ^- is required to satisfy

$$\left\langle \Phi_0^-, \mu^- \right\rangle = \left\langle \Phi_0^-, \bar{\mu}^- \right\rangle = 0\,, \qquad \frac{1}{4}\mathcal{J}^-_{0\,AB}\Gamma^{AB}\mu^- = \mathrm{i}\mu^-\ . \tag{3.29}$$

Under the SU(3,3) group defined by Φ_0^-, the $\mathbf{32}'$ spinor representation decomposes as $\mathbf{1} + \mathbf{1} + \mathbf{15} + \bar{\mathbf{15}}$. The conditions (3.29) imply that μ^- is in the $\mathbf{15}$ representation, and hence we see there are a further 30 real elements in $\mathfrak{e}_{7(7)}$ which stabilise the $K_a^{(0)}$. Thus together with the $\mathfrak{su}(3,3)$ algebra and the element (3.28) we see that the stabiliser group is 66 dimensional. It is relatively straightforward to show that this group has signature (30, 36) and hence corresponds to $\mathrm{SO}^*(12)$.

We now address how to generate a generic element in the orbit from the specific $K_a^{(0)}$ discussed so far. We first note that the $O(6,6) \subset E_{7(7)}$ transformations of Φ_0^- by B as in (2.4) generate the full $O(6,6)$ orbit. These transformations are embedded in $E_{7(7)}$ as in (3.13). On the other hand, $\mathrm{SL}(2,\mathbb{R})$ elements simply rotate u^i, which was already assumed to be general. Apart from the SU(2) rotations among the $K_a^{(0)}$, the only additional motion in the orbit comes from elements of the form $(0,0,\mu^{i-})$. We expect that these should correspond to the RR scalars C^-. To see this explicitly, we first recall that 30 of these leave $K_a^{(0)}$ invariant, while anything of the form $\mu^{i-} = A\,\mathrm{Re}(u^i\Phi^-) + B\,\mathrm{Im}(u^i\Phi^-)$ simply generates part of the SU(2) rotations among the triplet $K_a^{(0)}$. The remaining 32 degrees of freedom can be generated by acting with the RR potential C^- embedded in $E_{7(7)}$ as in (3.13), since it is easy to show that none of these elements leave $K_a^{(0)}$ invariant. Hence the generic triplet in the orbit can be written as

$$K_a = \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}K_a^{(0)\prime}\,, \tag{3.30}$$

where $K_a^{(0)\prime}$ are the SU(2) rotation of $K_a^{(0)}$, that we can parameterise as

$$\begin{aligned}
K_3^{(0)\prime} &= \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\alpha}\,K_+^{(0)} + \frac{1}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}\alpha}\,K_-^{(0)} - \cos\theta\,K_3^{(0)}\,,\\
K_+^{(0)\prime} &= \frac{1}{2}(1-\cos\theta)\mathrm{e}^{\mathrm{i}(\psi+\alpha)}K_+^{(0)} - \frac{1}{2}(1+\cos\theta)\mathrm{e}^{\mathrm{i}(\psi-\alpha)}K_-^{(0)} - \mathrm{e}^{\mathrm{i}\psi}\sin\theta K_3^{(0)}\ .
\end{aligned} \tag{3.31}$$

Note that the angle α corresponds to an $\mathrm{U}(1) \subset \mathrm{SU}(2)$ phase rotation on $K_+^{(0)}$, which can be absorbed in Φ_0^-. Similarly the $\mathrm{e}^{\tilde{B}}$ action is in $\mathrm{SL}(2,\mathbb{R})$ and so is strictly speaking unnecessary since it can be absorbed in a redefinition of u^i. However it is useful to include it to see the structure of how the supergravity potentials appear. To see that the orbit is

68-dimensional we note that Φ^- and C^- each contributes 32 degrees of freedom. In the original ansatz we can always rescale $\Phi^- \to c\Phi^-$ and $u^i \to c^{-1}u^i$ for $c \in \mathbb{C} - \{0\}$ so there are really only two new real degrees of freedom in u^i. In addition there are two degrees of freedom in θ and ψ giving a total of 68.

Having given an explicit parameterisation of the coset space, we can now consider the hyperkähler structure and hyperkähler potential following ref. [35]. The result for the latter is very simple: at a generic point on $\mathcal{M}_{\rm HKC}$ it is given by

$$\chi = \sqrt{-\frac{1}{8}\,\mathrm{tr}(K_+K_-)}\,, \tag{3.32}$$

where the trace is defined in (3.7). Inserting (3.25), (3.30) and (3.31) we find that

$$\chi = \sqrt{\frac{1}{8}(-\mathrm{i}u\bar{u})\mathrm{i}\langle\Phi^-,\bar{\Phi}^-\rangle} = \sqrt{\frac{1}{8}(-\mathrm{i}u\bar{u})H(\mathrm{Re}\,\Phi^-)}\,, \tag{3.33}$$

where in the second equation we have used (2.13) and (2.20). Note that in fact $\chi = \frac{1}{2}\kappa$ where κ was the normalization of the $\mathfrak{su}(2)$ algebra (3.27) and so is manifestly an SU(2) invariant. Furthermore, it is independent of C^- since it is an $E_{7(7)}$-invariant function of K_a, and e^{C^-} is an $E_{7(7)}$ transformation. Comparing with (3.21) we see that the χ agree if we identify[17]

$$-\mathrm{i}u\bar{u} = 2\mathrm{e}^{-2\phi}\,. \tag{3.34}$$

In section 4.1 we compute the Killing prepotentials which will allow us to determine the dilaton dependence of u, providing an independent confirmation of (3.34).

In addition to the hyperkähler potential there should be a triplet of complex structures (I_1, I_2, I_3) acting on vectors in the tangent space of the cone. Recall that a general point on the cone is defined by the triplet K_a, while a generic vector can be viewed as small deformation δK_a along the cone. A general deformation around the orbit is generated by the action of some $\mu \in \mathbf{133}$ on K_a. To fill out the full cone we also need to consider rescalings of K_a. Thus a vector in the tangent space of the cone at the point K_a is a triplet that can be written as

$$\xi_a = [\mu, K_a] + \mu_0 K_a. \tag{3.35}$$

for some $\mu \in \mathbf{133}$ and $\mu_0 \in \mathbb{R}^+$. Since K_a satisfy the $\mathfrak{su}(2)$ algebra (3.27), one only needs to specify two elements (say K_1 and K_2 or equivalently K_+) to determine the triplet. Similarly, the vector in the tangent space is completely determined by giving only two of the three ξ_a. The three complex structures are then most easily defined by picking out different pairs of ξ_a to specify the vector. In particular one defines the structure I_3 by taking the vector defined by the pair (ξ_1, ξ_2) with the simple action

$$I_3\,(\xi_1 + \mathrm{i}\xi_2) = \mathrm{i}\,(\xi_1 + \mathrm{i}\xi_2)\,, \tag{3.36}$$

with the corresponding cyclic relations defining I_1 and I_2.

[17] In a generic SU(2) gauge, i.e. using χ as found in [37], we would get $-\mathrm{i}u\bar{u} = 2\mathrm{e}^{-2\phi}\sin^2\theta$. This would not spoil the consistency verified in section 4.1 below, so for convenience we use the gauge fixed expression (3.21). We thank B. Pioline for discussions on this point.

3.4 Vector multiplets

We now turn to the vector multiplet moduli space $\mathcal{M}^+_{\text{SK}}$. The superconformal supergravity formalism requires an additional vector multiplet whose scalar degrees of freedom are the conformal compensator corresponding to the overall scale of Φ^+. This adds a $\text{U}(1) \times \mathbb{R}^+$ factor to the moduli space, turning the local special Kähler geometry $\mathcal{M}^+_{\text{SK}}$ into a rigid one $\widetilde{\mathcal{M}}^+_{\text{SK}}$. Both of them are expected to be cosets of the form $E_{7(7)}/H$, up to an $\mathbb{R}^+$ factor. In the following section we anticipate the form of the coset. In section 3.4.2 we identify the embedding of the NS degrees of freedom into an orbit which spans the expected coset, and show that the Kähler potential is given by the square-root of the $E_{7(7)}$ quartic invariant in complete analogy to the Hitchin function in the $O(6,6)$ case.

3.4.1 Expected cosets for vector multiplet moduli space

It is well known [43] which coset manifolds have a local special Kähler geometry and there is only one candidate based on $E_{7(7)}$

$$\mathcal{M}^+_{\text{SK}} = \frac{E_{7(7)}}{E_{6(2)} \times \text{U}(1)} . \tag{3.37}$$

(Again we are actually using a particular non-compact and non-Riemannian version with signature $(30, 24)$.) There is also the corresponding rigid special Kähler space

$$\widetilde{\mathcal{M}}^+_{\text{SK}} = \frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+ , \tag{3.38}$$

such that $\mathcal{M}^+_{\text{SK}} = \widetilde{\mathcal{M}}_{\text{SK}}/\mathbb{C}^*$.

3.4.2 Explicit construction

We would like to see explicitly how Φ^+ can be used to parameterise the special Kähler coset spaces (3.37) and (3.38) and how the metric on each is defined in terms of $E_{7(7)}$ objects.

The space $\widetilde{\mathcal{M}}^+_{\text{SK}}$ is actually what is known as a "prehomogeneous" vector space [45], that is, it is an open orbit of $E_{7(7)}$ in the 56-dimensional representation. This is in complete analogy to the $\Phi^\pm$ moduli spaces, which were open orbits in the spinor representations $\mathbf{32}^\pm$ under $O(6,6)$. However, it is in contrast to the hypermultiplet space $\mathcal{M}_{\text{HKC}}$ discussed above. For us, the main point is that we should be able to realise the space as the orbit of some embedding of Φ^+ in the $\mathbf{56}$ representation. As discussed in section 3.1, under $\text{SL}(2,\mathbb{R}) \times O(6,6)$, the fundamental representation decomposes as $\lambda = \left(\lambda^{iA}, \lambda^+\right)$ corresponding to $\mathbf{56} = (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32})$. We would like to have a real orbit, so it is natural to start with an embedding

$$\lambda^{(0)} = \left(0, \rho^+_0\right) , \tag{3.39}$$

where $\rho^+_0 = 2 \operatorname{Re} \Phi^+_0$. (The factor of two is chosen to match (2.20)).

To check that this is a reasonable choice, we first note that it implies that Φ^+ is a singlet under the S-duality group $\text{SL}(2,\mathbb{R}) \subset E_{7(7)}$. This is exactly what we would expect. In section 2 we recalled that the dilaton is part of a hypermultiplet, and therefore it should not couple to Φ^+, implying the latter is a singlet under S-duality. Alternatively, in type IIA the $\mathrm{U}(1)_\mathrm{R}$ acts on the Spin(6) spinors by the phase rotation $\eta^I_+ \to \mathrm{e}^{\mathrm{i}\alpha/2}\eta^I_+$. This follows from (2.2) together with the fact that the four-dimensional supersymmetry parameters ε^1_+, ε^2_+ transform with opposite phases under the $\mathrm{U}(1)_\mathrm{R}$, while the ten-dimensional ϵ^1, ϵ^2 are invariant.[18] Using (2.4) we see that Φ^+ is a singlet under the $\mathrm{U}(1)_\mathrm{R}$ in type IIA while Φ^- rotates with a phase. This phase rotation is generated by the $\mathrm{SU}(2)_R$ generator K_3 defined in section 3.3.3. Since (3.25) shows the embedding of $\mathrm{SU}(2)_\mathrm{R}$ into $E_{7(7)}$, we conclude that Φ^+ has to be singlet under the S-duality $\mathrm{SL}(2,\mathbb{R})$. Thus we see that embedding Φ^+ into the **56** of $E_{7(7)}$ is also consistent with the action of the $N=2$ R-symmetry.

In order to fill out the full orbit, we must act on $\lambda^{(0)}$ with $E_{7(7)}$ so

$$\lambda = g\cdot\lambda^{(0)}\,, \qquad g\in E_{7(7)}\ . \tag{3.40}$$

We can see that the dimension of the orbit is indeed 56 by looking at the stabiliser of $\lambda^{(0)}$. Using the decomposition (3.3) we see from (3.5) that $\delta\lambda^{(0)}=0$ holds for the following 78 elements of $e_{7(7)}$: three from the $\mathrm{SL}(2,\mathbb{R})$ elements $\mu^i{}_j$, which do not act on $\lambda_{(0)}$; 35 from $\mu^A{}_B$ since by construction $\mathrm{Re}\,\Phi^+_0$ is stabilised by $\mathrm{SU}(3,3)\subset O(6,6)$; and 40 from μ^{i-}, since they must satisfy $\left\langle\mu^{i-},\Gamma^A\,\mathrm{Re}\,\Phi^+_0\right\rangle=0$, giving 24 conditions for 64 parameters μ^{i-}. Put another way, decomposing under $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(3,3)\subset E_{7(7)}$, the adjoint action of the stabiliser group transforms as $(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{35})+(\mathbf{2},\mathbf{20})$, which is precisely how the adjoint of $E_{6(2)}$ decomposes under $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(3,3)\subset E_{6(2)}$.

Since the full orbit for K_a corresponded to a $\mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}$ transformation on $K^{(0)}_a$ we might expect the same for λ. That is, the generic element is given by

$$\lambda = \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}\lambda^{(0)}\ . \tag{3.41}$$

However, this is not yet the full story. Such transformations do not quite fill out the orbit. Instead, 12 degrees for freedom are missing. We will come back to this point in the following section.

As we have mentioned, it is well-known [42, 43, 46] that the space (3.38) admits a special Kähler metric. We now turn to the explicit construction of this metric, which will follow exactly Hitchin's construction of the corresponding special Kähler metric on the space $O(6,6)/\mathrm{SU}(3,3)\times\mathbb{R}^+$ of $\mathrm{Re}\,\Phi^\pm$ [7]. As there, we start with a natural symplectic structure, since, as discussed in section 3.1, by definition $E_{7(7)}$ preserves a symplectic structure $\mathcal{S}(\lambda,\lambda')$ on the fundamental representation. The complex structure, and hence special Kähler geometry then arise from the natural generalisation of the Hitchin function (2.20). Instead of the $O(6,6)$ spinor quartic invariant we take the quartic invariant $Q(\lambda)$ in (3.4) that defines the $E_{7(7)}$ group. We then define the Hitchin function

$$H(\lambda) = \sqrt{Q(\lambda)}\ . \tag{3.42}$$

As before, one can view $H(\lambda)$ as a Hamiltonian and define the corresponding Hamiltonian vector field $\hat{\lambda}$, given by, for any ν in the **56** representation,

$$\mathcal{S}(\nu,\hat{\lambda}) = -\nu^{\mathcal{A}}\frac{\partial H}{\partial\lambda^{\mathcal{A}}}\,, \tag{3.43}$$

[18] The same argument implies that in type IIB the spinors η^I_+ rotate with opposite phases.

where $\mathcal{A} = 1, \ldots, 56$ runs over the elements of the fundamental representation. Explicitly we have

$$
\begin{aligned}
\hat{\lambda}^{iA} &= \frac{1}{2H}\langle\lambda^+, \Gamma^A{}_B\lambda^+\rangle\lambda^{iB} - \frac{1}{H}\left(\lambda^i \cdot \lambda_j\right)\lambda^{jA}, \\
\hat{\lambda}^+ &= -\frac{1}{2H}\left(\frac{1}{12}\langle\lambda^+, \Gamma_{AB}\lambda^+\rangle - \epsilon_{ij}\lambda^i_A\lambda^j_B\right)\Gamma^{AB}\lambda^+ \, .
\end{aligned} \tag{3.44}
$$

The complex structure on $\widetilde{\mathcal{M}}^+_{\rm SK}$ is then given by

$$
J_{\rm SK}{}^{\mathcal{A}}{}_{\mathcal{B}} = \frac{\partial\hat{\lambda}^{\mathcal{A}}}{\partial\lambda^{\mathcal{B}}} \, . \tag{3.45}
$$

Equivalently the metric on $\widetilde{\mathcal{M}}^+_{\rm SK}$ is given by the Hessian

$$
g^{\rm SK}_{\mathcal{AB}} = \frac{\partial H}{\partial\lambda^{\mathcal{A}}\partial\lambda^{\mathcal{B}}} \, . \tag{3.46}
$$

Following the same arguments of [7] it is easy to show that this metric is special Kähler. Finally, note that one can define the holomorphic object, analogous to Φ^+,

$$
L = \frac{1}{2}(\lambda + {\rm i}\hat{\lambda}) \, , \tag{3.47}
$$

such that the Kähler potential is given by

$$
{\rm e}^{-K_{\rm SK}} = H(\lambda) = {\rm i}\mathcal{S}(L, \bar{L}) \, . \tag{3.48}
$$

On the subspace ${\rm e}^{-B}\lambda^{(0)} = (0, \rho^+)$ we have

$$
\begin{aligned}
H(\lambda) &= \sqrt{q(\rho^+)} \, , \\
L &= (0, \Phi^+) \, , \\
J_{\rm SK} \cdot \nu &= \left(\mathcal{J}^{-A}{}_B\nu^{iB}, J^+_{\rm Hit} \cdot \nu^+\right) ,
\end{aligned} \tag{3.49}
$$

where $q(\rho^+)$ is the spinor quartic invariant (2.21) and $J^+_{\rm Hit}$ is the Hitchin complex structure (2.9) on the spinor space. Thus we see that the special Kähler metric on $\mathbb{R}^+ \times E_{7(7)} \,/\, E_{6(2)}$ reduces to the special Kähler metric on $\mathbb{R}^+ \times O(6,6)/{\rm SU}(3,3)$ on this subspace.

3.5 Hypermultiplets and vector multiplets: compatibility conditions and SU(8) representations

In this section we turn to the question of compatibility between the structures arising in the vector multiplet and hypermultiplet sectors.

To start with, note that the vector multiplet moduli space $\mathcal{M}^+_{\rm SK}$ is 54-dimensional, whereas the original $O(6,6)/{\rm U}(3,3)$ space was 30-dimensional, and one would expect no additional degrees of freedom in this sector. A partial answer to this discrepancy is that, as in the $O(6,6)$ case, we expect there to be some compatibility condition between the hypermultiplet ${\rm SO}^*(12)$ structure and the vector multiplet $E_{6(2)}$ structure. This can be seen by considering the way the supergravity degrees of freedom are encoded in

$E_{7(7)}$. Recall that in the EGG the internal bosonic metric, form-field and axion-dilaton degrees of freedom are encoded in the exceptional generalised metric (3.14). This defines an $\mathrm{SU}(8)/\mathbb{Z}_2$ structure on the exceptional generalised tangent space E. The fermions in the supergravity transform under the local SU(8) group, as do the supersymmetry parameters. In particular, recall that for the $O(6,6)$ case the pair (η^1, η^2) transforms under $\mathrm{Spin}(6) \times \mathrm{Spin}(6) \simeq \mathrm{SU}(4) \times \mathrm{SU}(4)$. Thus to see the SU(8) transformation properties we simply rewrite our original spinor decomposition (2.2) as

$$\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} = \varepsilon^1_+ \otimes (\theta^1)^* + \varepsilon^2_+ \otimes (\theta^2)^* + \text{c.c.} \tag{3.50}$$

where $(\theta^I)^*$ are the complex conjugates of two elements θ^1 and θ^2 of the **8** representation of SU(8)

$$\theta^1 = \begin{pmatrix} \eta^1_+ \\ 0 \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} 0 \\ \eta^2_- \end{pmatrix} . \tag{3.51}$$

Together, the pair (θ^1, θ^2) is invariant under $\mathrm{SU}(6) \subset \mathrm{SU}(8)$ transformations. Thus we see that for $N = 2$ supersymmetry, comparing the generalised and exceptional generalised geometries we have the structures

$$\begin{array}{rlll} \text{gen. geom.:} & \mathrm{SU}(3) \times \mathrm{SU}(3) & \subset O(6) \times O(6) & \subset O(6,6) \\ \text{exceptional gen. geom.:} & \mathrm{SU}(6) & \subset \mathrm{SU}(8)/\mathbb{Z}_2 & \subset E_{7(7)} \ . \end{array} \tag{3.52}$$

Thus in general we expect the hypermultiplet $\mathrm{SO}^*(12)$ structure and the vector multiplet $E_{6(2)}$ structure to be constrained such that they have a common SU(6) subgroup, that is, as embedding in $E_{7(7)}$ compatibility requires

$$\mathrm{SO}^*(12) \cap E_{6(2)} = \mathrm{SU}(6) \ . \tag{3.53}$$

Thus together the consistent hypermultiplet and vector multiplet moduli spaces, coming from the cones $\mathcal{M}_{\mathrm{HKC}}$ and $\widetilde{\mathcal{M}}^+_{\mathrm{SK}}$, describe the coset space

$$\widetilde{\mathcal{M}} = \frac{E_{7(7)}}{\mathrm{SU}(6)} \times \mathbb{R}^+ \times \mathbb{R}^+ , \tag{3.54}$$

or if we go to $\mathcal{M}_{\mathrm{QK}}$ and $\mathcal{M}^+_{\mathrm{SK}}$, we have $E_{7(7)} / (\mathrm{SU}(6) \times \mathrm{U}(2))$. The U(2) factor corresponds to R-symmetry rotations on the two θ^1 and θ^2. This last space is 94-dimensional with signature $(70, 24)$, while the sum of the dimensions of $\mathcal{M}_{\mathrm{QK}}$ and $\mathcal{M}_{\mathrm{SK}}$ is 118. This implies that the compatibility condition (3.53) should impose 24 conditions. Let us see if this is indeed the case: requiring that the $SO^*(12)$ stabiliser of K_a shares a common SU(6) subgroup with the $E_{6(2)}$ stabiliser of λ translates into the requirement

$$K_a \cdot \lambda = 0 , \qquad a = 1, 2, 3 , \tag{3.55}$$

where we are simply taking the adjoint action on the fundamental representation. It is equivalent to $K_+ \cdot L = 0$ (with L defined in (3.47)). In particular we see from (3.5) and (3.25) that

$$\begin{aligned}\left(K_+^{(0)}\cdot L^{(0)}\right)^{iA} &= u^i\left\langle\Phi^-,\Gamma^A\Phi^+\right\rangle\,,\\ \left(K_+^{(0)}\cdot L^{(0)}\right)^{+} &= 0\,,\end{aligned}\tag{3.56}$$

so, at this point, compatibility is equivalent to the compatibility condition (2.10) between Φ^+ and Φ^-, which amounts only to 12 conditions. Thus there are 12 conditions unaccounted for. On the other hand, if we count up the degrees of freedom in Φ^+,Φ^-, C^- and $\phi,\tilde{B}$ we get $48+32+2=82$, while $\widetilde{\mathcal{M}}$ is 94-dimensional, again leaving 12 degrees of freedom unaccounted for.

Looking at our spinor ansatz (3.51) we can immediately see what is missing: the expressions for θ^1 and θ^2 are not generic. A generic SU(6) structure is given by

$$\theta^1=\begin{pmatrix}\eta_+^1\\ \tilde{\eta}_-^1\end{pmatrix}\,,\qquad\qquad \theta^2=\begin{pmatrix}\tilde{\eta}_+^2\\ \eta_-^2\end{pmatrix}\,.\tag{3.57}$$

$\tilde{\eta}^1,\tilde{\eta}^2$ introduce 16 real new parameters. However, there is a U(2) R-symmetry rotating the two θ^I, which can be used to remove four parameters, and therefore there are indeed precisely 12 new degrees of freedom. The special property of the ansatz (3.51) is that the SU(6) structure decomposes into $\mathrm{SU}(3)\times\mathrm{SU}(3)$ under $O(6,6)\subset E_{7(7)}$. There is nothing about this freedom that is special to the $E_{7(7)}$ formulation, we could always have used it to generalise our original $N=2$ ansatz (2.2) even in the $O(6,6)$ formulation. In terms of the SU(6) subgroup, these extra degrees of freedom transform in the $\mathbf{6}+\bar{\mathbf{6}}$ representation. Including these degrees of freedom, as we have mentioned, the "local" version of $\widetilde{\mathcal{M}}$, i.e. the coset $E_{7(7)}\,/(\mathrm{SU}(6)\times\mathrm{U}(2))$ contains 24 non-physical modes beyond the 70 parameterised by the supergravity degrees of freedom g, B, C^- and $(\phi,\tilde{B})$. As before, we expect that these are related to the massive spin-$\frac{3}{2}$ degrees of freedom and can be gauged away. Alternatively we can view this as simply projecting out all the $\mathbf{6}$ and $\bar{\mathbf{6}}$ degrees of freedom.

Regarding the number of conditions imposed by compatibility, we saw in the previous section that the generic element $\lambda=e^{C^-}\lambda^{(0)}$ has 44 degrees of freedom and therefore the action of C^- does not fill out the full 56-dimensional orbit $\widetilde{\mathcal{M}}^+_{\mathrm{SK}}$. The missing 12 extra degrees of freedom are precisely those that correspond to using the generic spinor ansatz (3.57). Thus we can write the generic element λ in the form (3.41), provided $\lambda^{(0)}=\tilde{g}(0,\rho^+)$ where $\tilde{g}\in\mathrm{SU}(6)\subset\mathrm{SU}(8)/\mathbb{Z}_2\subset E_{7(7)}$ is the element which transforms the restricted ansatz (3.51) to the general form (3.57). Equivalently we can write λ in the terms of the $\mathrm{SU}(8)/\mathbb{Z}_2$ decomposition (3.8) of $E_{7(7)}$, as

$$\begin{aligned}\lambda=(\lambda^{\alpha\beta},\lambda_{\alpha\beta}) &= \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B}\lambda^{(0)}\,,\\ \lambda^{(0)\alpha\beta}=\epsilon_{IJ}\theta^{I\alpha}\theta^{J\beta}\,, &\qquad\qquad \bar{\lambda}^{(0)}_{\alpha\beta}=\epsilon_{IJ}\theta^{I*}_\alpha\theta^{J*}_\beta\,,\end{aligned}\tag{3.58}$$

where $\epsilon_{12}=-\epsilon_{21}=1$ and θ^I are the generic spinors (3.57). In this form, it is easy to see that λ is invariant under $\mathrm{SU}(2)_{\mathrm{R}}$. For a generic element in the **56**, compatibility requirement (3.55) amounts to 24 conditions, as opposed to 12 for the case of λ belonging to the 44-dimensional orbit.

From a supergravity perspective the appearance of C^- in the vector multiplet sector is unexpected, since in simple Calabi-Yau models it corresponds to a hypermultiplet degree of freedom. Equally odd is that the moduli space $O(6,6)/\mathrm{SU}(3,3)\times\mathbb{R}^+$ spanned by Φ^+ has been promoted to the larger space (3.38). As we have discussed these extra degrees of freedom can be accounted for by compatibility condition (3.55) and the generalised spinor ansatz (3.57). It is helpful to note, however, that if we project out all the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ triplets (or equivalently the SU(6) representations $\mathbf{6}$ and $\bar{\mathbf{6}}$), then the moduli spaces of λ and Φ^+ agree, and C^- does not appear in the vector multiplet sector. As we argued, projecting out the triplets precisely ensures the absence of additional (massive) spin-$\frac{3}{2}$ gravitino multiplets. Their presence would change the standard form of the $N=2$ supergravity and in particular the decoupling of vector multiplets and hypermultiplets would no longer hold. The construction presented here shows that if we include all representations then we can rewrite the field space in terms of $E_{7(7)}$ objects, but with vector multiplet and hypermultiplet moduli spaces coupled through the compatibility condition (3.55). The expectation is that the additional (non-physical) coupled degrees of freedom are associated to the massive spin-$\frac{3}{2}$ multiplets and can be gauged away.

In the previous sections, we actually always used the restricted $\mathrm{SU}(3)\times\mathrm{SU}(3)$ spinor ansatz given by (3.51) when making explicit calculations. This means that we do not quite fill out the true moduli spaces. Nonetheless our final expressions are written as though all the structures were generic, so that, although we calculated in a slightly restricted case, we believe the resulting formulas are in fact true in general. One key point, as we will see, is that the ansatz (3.51) is general enough to encode the generic supersymmetric $N=1$ vacua.

Let us end this section by noting that the coset space $\mathcal{M}_{\mathrm{QK}}$ describing the hypermultiplet moduli space can also be simply described in terms of the $\mathrm{SU}(8)/\mathbb{Z}_2$ decomposition of $E_{7(7)}$ directly as bilinears of the SU(8) spinors θ^I. Using the notation of (3.8) we have

$$\begin{aligned} K_a &= (K_a{}^\beta{}_\beta, K_a^{\alpha\beta\gamma\delta}, \bar{K}_{a\,\alpha\beta\gamma\delta}) = \mathrm{e}^{C^-}\,\mathrm{e}^{\tilde{B}}\,\mathrm{e}^{-B}\,\mathrm{e}^{-\phi}K_a^{(0)} \\ K_a^{(0)\alpha}{}_\beta &= \frac{1}{2}\sigma_{a\,I}{}^J\theta^{I\,\alpha}\bar{\theta}_{J\,\beta}\,, \qquad K_a^{(0)\alpha\beta\gamma\delta} = 0\,, \qquad \bar{K}^{(0)}_{a\,\alpha\beta\gamma\delta} = 0\,, \end{aligned} \tag{3.59}$$

where σ_a are the Pauli matrices, e^{C^-} is the action of the RR scalars C^- in $E_{7(7)}$ as above, $\mathrm{e}^{\tilde{B}}$ is the axion action in $\mathrm{SL}(2,\mathbb{R})\subset E_{7(7)}$ and e^{-B} is just the usual action of the NS B-field in $O(6,6)$ embedded in $E_{7(7)}$. Here we span the full 63-dimensional subspace just using the restricted ansatz (3.51). This reproduces (3.25) in the gauge (3.34). From (3.50) we see that the $\mathrm{SU}(2)_\mathrm{R}$ R-symmetry acts on the doublet θ^I. Given the form (3.59), this translates into rotations of the triplet K_a as expected. It is also easy to check compatibility with λ, namely using (3.58) and (3.9) we see that $K_a\cdot\lambda = 0$.

4 Killing prepotentials and $N=1$ vacua

Thus far we have identified how the vector multiplet and hypermultiplet degrees of freedom are naturally encoded as orbits in the fundamental and adjoint representations of $E_{7(7)}$ respectively. In particular we have identified the corresponding special Kähler and quaternionic geometries which govern the kinetic terms of the fields.

In this section we turn first to the Killing prepotential terms in the $N = 2$ action, which encode the gauging of the vector multiplets and the scalar potential, and second, we briefly discuss the form of the $N = 1$ supersymmetric vacua equations in this formulation. Both objects, unlike the kinetic terms, now depend on the differential structure of the EGG, but, as we will see, can still be written in $E_{7(7)}$ form.

4.1 Killing prepotentials

Since we are interested in the differential structure, we start by introducing an embedding of the exterior derivative d in $E_{7(7)}$. Taking a slightly different version of the EGT, the one weighted by $(\Lambda^6 T^*M)^{-1/2}$ (see (A.5), (A.6)), namely

$$E_{-1/2} = TM \oplus T^*M \oplus \Lambda^5 TM \oplus \big(TM \otimes \Lambda^6 TM\big) \oplus \Lambda^{\text{even}} TM\,, \qquad (4.1)$$

we then embed the exterior derivative d in the one-form component T^*M. In the notation of eq. (3.2) this defines an element of the **56**

$$D = (D^{iA}, D^+) = (v^i \mathrm{d}^A, 0)\,, \qquad A = 1, \ldots, 12\,, \qquad (4.2)$$

where the operator d^A only has entries in its 'lower' six components, i.e. $\mathrm{d}^A = (0, \partial_m)$ where $m = 1, \ldots, 6$.

The form of the $N = 2$ prepotentials (2.22) and (3.22) suggests that the Killing prepotentials on the hyperkähler cone can be written in terms of $E_{7(7)}$ objects as

$$\mathcal{P}_a^{\text{HKC}} = \mathrm{i}\mathcal{S}(L, DK_a)\,, \qquad (4.3)$$

where the symplectic pairing $\mathcal{S}$ is defined in (3.4) and DK_a represents the **56** component in the product **56** $\times$ **133**, that is, the usual action of the adjoint on the fundamental representation. Let us show that this is indeed the case. We will give the proof for the slightly restricted ansatz (3.51). However, given it can be put in $E_{7(7)}$ form, we believe it to be true in general.

We first note that the compatibility conditions (3.55) imply that an SU(2) rotation on K_a simply rotates the prepotentials $\mathcal{P}_a$ as expected. In particular the terms with derivatives of the rotation matrix drop out. Thus in calculating (4.3) we can effectively take $K_a^{(0)\prime} = K_a^{(0)}$ in (3.31), and hence $K_a = \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B} K_a^{(0)}$. We also have $L = \mathrm{e}^{C^-}\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B} L_0$ so we can rewrite $\mathcal{S}(L, DK_a) = \mathcal{S}(\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B} L_0, \mathrm{e}^{-C^-} DK_a)$. The explicit calculation of $\mathrm{e}^{-C^-} DK_a$ is given in appendix B. Using $\mathrm{e}^{\tilde{B}}\mathrm{e}^{-B} L_0 = (0, \Phi^+)$ we then find

$$\begin{aligned}
\mathcal{P}_+^{\text{HKC}} &= \mathrm{i}\mathcal{S}(L, DK_+) = \mathrm{i}(uv)\big\langle \Phi^+, \mathrm{d}\Phi^- \big\rangle\,, \\
\mathcal{P}_-^{\text{HKC}} &= \mathrm{i}\mathcal{S}(L, DK_-) = \mathrm{i}(\bar{u}v)\big\langle \Phi^+, \mathrm{d}\bar{\Phi}^- \big\rangle\,, \\
\mathcal{P}_3^{\text{HKC}} &= \mathrm{i}\mathcal{S}(L, DK_3) = -\mathrm{i}\frac{(uv)(\bar{u}v)}{\sqrt{-2\mathrm{i}u\bar{u}}} \sqrt{\mathrm{i}\big\langle \Phi^-, \bar{\Phi}^- \big\rangle}\,\big\langle \Phi^+, \mathrm{d}C^- \big\rangle\,.
\end{aligned} \qquad (4.4)$$

The next step is to compare these expressions with (3.22) and (2.22). We find

$$uv = -2\mathrm{e}^{\frac{1}{2}K^- + \phi}\chi\, \mathrm{e}^{\mathrm{i}\alpha}\,, \qquad \frac{(uv)(\bar{u}v)}{\sqrt{-2\mathrm{i}u\bar{u}}} \sqrt{\mathrm{i}\big\langle \Phi^-, \bar{\Phi}^- \big\rangle} = \mathrm{e}^{2\phi}\chi\,. \qquad (4.5)$$

Inserting the first equation into the second and using (2.13), we confirm eq. (3.33) for the the hyperkähler potential, and thus are left with

$$uv = -\sqrt{-\frac{1}{2}\mathrm{i}u\bar{u}\,\mathrm{e}^{\phi}\mathrm{e}^{\mathrm{i}\alpha}}\ . \tag{4.6}$$

We have already argued that the conformal compensator degrees of freedom Y correspond to a common rescaling of u^i. Using a parameterisation adapted to the convention of appendix A where we take $v^1 = 1$ and $v^2 = 0$, we write [19]

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = Y \begin{pmatrix} S \\ -1 \end{pmatrix} \ . \tag{4.7}$$

Since u^i transforms as an $\mathrm{SL}(2,\mathbb{R})$ doublet according to

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \ , \qquad ad - bc = 1\,, \tag{4.8}$$

one checks that S indeed transforms by fractional linear transformations

$$S \rightarrow -\frac{aS - b}{cS - d}\ . \tag{4.9}$$

Inserting (4.7) (or its covariant version, as in footnote 19) into (4.6) yields

$$Y = r\mathrm{e}^{\mathrm{i}\alpha} = -uv\ , \qquad S - \bar{S} = 2\mathrm{i}\mathrm{e}^{-2\phi}\ . \tag{4.10}$$

As anticipated the comparison of the Killing prepotentials successfully determined the dilaton dependence in u^i. Inserting back (4.10) into (3.33) we indeed find the hyperkähler potential χ as given in (3.21), where, in the $O(6,6)$ version, one uses the rescaling ambiguity between u^i and Φ^- mentioned below (3.31) to interpret $\sqrt{r}$ as the overall scale of Φ^-. We also note that, as expected we can generate the axion component of S by the $\tilde{B}$-transformation $\mathrm{SL}(2,\mathbb{R})$ action given in (3.13). Explicitly, if we get the map

$$u^{(0)} = Y \begin{pmatrix} \mathrm{i}\mathrm{e}^{-2\phi} \\ -1 \end{pmatrix} \mapsto (\mathrm{e}^{\tilde{B}})^i{}_j u^{(0)\,j} = Y \begin{pmatrix} \mathrm{i}\mathrm{e}^{-2\phi} \\ -1 \end{pmatrix} + Y \begin{pmatrix} \tilde{B} \\ -1 \end{pmatrix} = Y \begin{pmatrix} S \\ -1 \end{pmatrix}\ , \tag{4.11}$$

where $S = \tilde{B} + \mathrm{i}\mathrm{e}^{-2\phi}$.

4.2 $N = 1$ vacuum equations

We now turn to a brief discussion of how the equations defining the $N = 1$ vacua for type II compactifications [6] can be reformulated in $E_{7(7)}$ language. We will simply give a sketch of the form of the corresponding equations leaving a full analysis for future work. Specifically we give $E_{7(7)}$ expressions which encode the $N = 1$ equations in their $O(6,6)$ spinor components, and then discuss to what extent these hold in general.

[19] Introducing an orthogonal vector ω^i (such that $v\omega = 1$) we can also write more covariantly, $u = Y(Sv + \omega)$.

Recall that defining an $N=1$ vacuum picks out a particular preserved supersymmetry in the $N=2$ effective theory breaking the $\mathrm{SU}(2)_R$ symmetry to $\mathrm{U}(1)_R$. Correspondingly, as discussed in [10, 11], one can identify the $N=1$ superpotential W as a complex linear combination of the $N=2$ Killing prepotentials $\mathcal{P}_a$. The remaining orthogonal combination is then related to the $N=1$ D-term. Concretely, given the decomposition (3.50), one identifies the preserved $N=1$ supersymmetry as $\varepsilon = \bar{n}^I \varepsilon_I$ where n_I is a normalised $\mathrm{SU}(2)_R$ doublet $\bar{n}^I$ (satisfying $\bar{n}^I n_I = 1$). Writing $n^1 = a$ and $n^2 = \bar{b}$, one then has the $\mathrm{U}(1)_R$ doublet and singlet combinations

$$\begin{aligned} \mathrm{e}^{K/2} W &= \mathrm{e}^{K^+/2} w^a \mathcal{P}_a, & (w^+, w^-, w^3) &= \left(a^2, -\bar{b}^2, -2a\bar{b}\right), \\ \mathcal{D} &= r^a \mathcal{P}_a. & (r^+, r^-, r^3) &= \left(ab, \bar{a}\bar{b}, |a|^2 - |b|^2\right), \end{aligned} \tag{4.12}$$

corresponding to the superpotential and D-term respectively.[20]

In general the equations governing $N=1$ vacua [6] should be obtainable by extremising the superpotential and setting the D-term to zero [16, 17]. Using the parameterisation above, the preserved supersymmetries take the form

$$\epsilon = \varepsilon_+ \otimes \mathrm{e}^{A/2}\, \theta^* + \text{c.c.}\,, \qquad \text{where} \qquad \theta = \begin{pmatrix} a\eta^1_+ \\ \bar{b}\eta^2_- \end{pmatrix}, \tag{4.13}$$

and A is the warp factor in front of the four-dimensional metric. In the special case of $|a|^2 = |b|^2$, and for zero cosmological constant on the four-dimensional space, the corresponding equations, in our conventions, are [16, 47]

$$\begin{aligned} \mathrm{d}\left(\mathrm{e}^{3A-\varphi}\Phi'^+\right) &= 0\,, \\ \left[\mathrm{d}\left(\mathrm{e}^{-\varphi}\,\mathrm{Re}\,\Phi'^-\right) - \mathrm{i}G\right]\big|_{1,0} &= 0\,, \\ \mathrm{d}\left(\mathrm{e}^{2A-\varphi}\,\mathrm{Im}\,\Phi'^-\right) &= 0\,. \end{aligned} \tag{4.14}$$

where

$$\Phi'^+ = a\bar{b}\Phi^+\,, \qquad \Phi'^- = ab\Phi^-\,, \tag{4.15}$$

while φ is the ten-dimensional dilaton and $|_{1,0}$ represents the projection onto the $+\mathrm{i}$ eigenspace using the Hitchin complex structure J^+_{Hit}. From (4.12) it is not hard to see that the first two equations in (4.14) can essentially be obtained from variations of the superpotential while the third one corresponds to the D-term equation.

An important point here is that although our original parameterisation of the $N=2$ supersymmetries was not completely generic, the $N=1$ vacuum parameterisation is generic. Recall that the $N=2$ $\mathrm{SU}(8)$ spinor ansatz given in (3.50) is restricted since some components vanish. Nonetheless, the $N=1$ supersymmetry (4.13) can be written with this ansatz as $a\theta^1 + \bar{b}\theta^2$ and is completely generic. The stabiliser of θ is $\mathrm{SU}(7)$. Thus,

[20] This is equivalent to making an $\mathrm{SU}(2)_R$ rotation by the matrix

$$M^I{}_J = \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1\,.$$

in terms of structures, since the $N = 1$ vacuum is determined solely by θ, it defines a particular SU(7) structure on the exceptional generalised tangent space E. By contrast, as we have seen, to define the $N = 2$ effective theory, we require two spinors θ^1 and θ^2 defining a SU(6) structure.

In the language of $E_{7(7)}$, the preserved $N = 1$ symmetry picks a particular complex structure on the hyperkähler cone, and uses it to define the $N = 1$ chiral field inside the hypermultiplet. In terms of ω^a, r^a, this reads[21]

$$K_3^{(0)\prime} = r^a K_a^{(0)}\,, \qquad K_+^{(0)\prime} = w^a K_a^{(0)}\,, \tag{4.16}$$

where K_3' corresponds to the particular $N = 1$ complex structure and K_+' defines the chiral field. Parameterising a and b as

$$a = \sin\frac{1}{2}\theta \mathrm{e}^{\mathrm{i}\gamma}\,, \qquad b = \cos\frac{1}{2}\theta \mathrm{e}^{\mathrm{i}\beta}\,, \tag{4.17}$$

and defining $\psi \equiv \gamma - \beta, \alpha \equiv \gamma + \beta$, we get $r^a K_a^{(0)} = K_3^{(0)'}$ then matches the expressions (3.31). As mentioned before, $\gamma + \beta$ can be identified with the $\mathrm{U}(1)_R$ angle α in the compensator Y (see eq. 4.10), while θ and $\gamma - \beta$ are the Euler angles. From the expression (4.15) for Φ'^+ we see that there should be a rescaling of L by $a\bar{b}$. Given the SU(8) covariant expression (3.58) we see that the phase of this rescaling corresponds to diagonal U(1) in the $\mathrm{U}(2)_R$ symmetry given by $\theta^I \mapsto \mathrm{e}^{\mathrm{i}(\gamma-\beta)/2}\theta^I$.

For the case $|a|^2 = |b|^2 = 1$ corresponding to $\theta = -\pi/2$, given we can always absorb two phases by redefining η^1 and η^2, without loss of generality we can set $\gamma + \beta = \alpha = \pi/2$, and $\psi = \pi/2$, so that

$$K_3^{(0)\prime} = K_2^{(0)}\,, \qquad K_+^{(0)\prime} = -K_1^{(0)} + \mathrm{i}K_3^{(0)}\ . \tag{4.18}$$

By comparing the expressions (3.25) and (3.49) for $K_a^{(0)\prime}$ and L one is led the naive conjecture for the generic $E_{7(7)}$ form of the $N = 1$ equations

$$D\left(\mathrm{e}^{3A-\varphi}L\right) = 0\,, \tag{4.19}$$

$$DK_+|_{1,0} = 0\,, \tag{4.20}$$

$$D\left(\mathrm{e}^{2A}K_3\right) = 0\ . \tag{4.21}$$

Here $K_a = \mathrm{e}^{C^-}\mathrm{e}^{-B}K_a^{(0)\prime}$ and the projector $|_{1,0}$ now projects onto the $+\mathrm{i}$ eigenspace of J_{SK}, the complex structure on the **56** representation defined by L and given in (3.45). Furthermore, in the first line we are taking the projection onto the **133** representation of $DL \in \mathbf{56} \times \mathbf{56}$. Finally we also choose $Y = \mathrm{e}^{-\varphi}$ in the expression (4.7) for u^i.

To investigate to what extent these equations hold, we again focus on the simple $|a|^2 = |b|^2$ case. Consider first the equation (4.19) for L. We need the projection onto the

[21] One can also derive these expressions from (3.59), rotating the SU(8) spinors θ^I by the matrix M given in footnote 20.

133 representation of the symmetric **56** $\times$ **56** tensor product. It is given by

$$\begin{aligned}
(\lambda\cdot\lambda')^{i}{}_{j} &= 2\epsilon_{jk}(\lambda^i\cdot\lambda'^k)\,,\\
(\lambda\cdot\lambda')^{A}{}_{B} &= 2\epsilon_{ij}\big[(\lambda^{iA}\lambda'^{j}{}_{B})+(\lambda'^{iA}\lambda^{j}{}_{B})\big]+\big\langle\lambda^+,\Gamma^{A}{}_{B}\lambda'^+\big\rangle\,,\\
(\lambda\cdot\lambda')^{i-} &= \left(\lambda^{iA}\Gamma_A\lambda'^+ + \lambda'^{iA}\Gamma_A\lambda^+\right)\,.
\end{aligned}\tag{4.22}$$

Writing $L = \mathrm{e}^{C^-}\mathrm{e}^{-B}L_0 = \mathrm{e}^{C^-}(0,\Phi^+)$, using the same generalised connection as in appendix B one finds

$$\begin{aligned}
\mathrm{e}^{-C^-}D\left(\mathrm{e}^{3A-\varphi}L\right)^{i}{}_{j} &= \sqrt{2}\mathrm{e}^{3A-\varphi}v^iv_j\big\langle\Phi^+,G\big\rangle\,,\\
\mathrm{e}^{-C^-}D\left(\mathrm{e}^{3A-\varphi}L\right)^{A}{}_{B} &= 0\,,\\
\mathrm{e}^{-C^-}D\left(\mathrm{e}^{3A-\varphi}L\right)^{i-} &= v^i\mathrm{d}\left(\mathrm{e}^{3A-\varphi}\Phi^+\right)\,,
\end{aligned}\tag{4.23}$$

where we have used $G=\sqrt{2}\mathrm{d}C^-$. We see that the spinor component $D(\mathrm{e}^{3A-\varphi}L)^{i-}=0$ indeed reproduces the first equation (4.14). The other components vanish provided $\langle\Phi^+,G\rangle=0$, but it is easy to see that this is implied by the first equation in (4.14).

Let us now turn to the hypermultiplet equations. Using the results in appendix B and the relations (4.18) we find for the spinor components

$$\begin{aligned}
\mathrm{e}^{-C^-}D\left(\mathrm{e}^{2A}K_3\right)^+ &= \mathrm{d}\left[e^{2A}(uv)\,\mathrm{Im}\,\Phi^-\right]\,,\\
\mathrm{e}^{-C^-}\left(DK_+\right)^+ &= -\mathrm{d}\left[(uv)\,\mathrm{Re}\,\Phi^-\right]-\mathrm{i}\frac{(uv)(\bar{u}v)}{\sqrt{-2\mathrm{i}u\bar{u}}}\sqrt{\mathrm{i}\big\langle\Phi^-,\bar{\Phi}^-\big\rangle}\mathrm{d}C^-
\end{aligned}\tag{4.24}$$

Given $Y=\mathrm{e}^{-\varphi}$ we have

$$uv=-\mathrm{e}^{-\varphi}\,,\qquad -\mathrm{i}u\bar{u}=2\mathrm{e}^{-2\varphi}\mathrm{e}^{-2\phi}=2\mathrm{e}^{-4\varphi}\,\mathrm{vol}_6\,,\tag{4.25}$$

where we have used (2.18). Since in our conventions, $\mathrm{i}\big\langle\Phi^-,\bar{\Phi}^-\big\rangle = 8\,\mathrm{vol}_6$ (see for instance (2.19)) we then have

$$\begin{aligned}
\mathrm{e}^{-C^-}D\left(\mathrm{e}^{2A}K_3\right)^+ &= -\mathrm{d}\left(\mathrm{e}^{2A-\varphi}\,\mathrm{Im}\,\Phi^-\right)\,,\\
\mathrm{e}^{-C^-}\left(DK_+\right)^+ &= \mathrm{d}\left(\mathrm{e}^{-\varphi}\,\mathrm{Re}\,\Phi^-\right)-\mathrm{i}G,
\end{aligned}\tag{4.26}$$

On the subspace $L=\mathrm{e}^{-B}L^{(0)}=(0,\Phi^+)$, by equation (3.49), we have for the spinor component of the **56** that $(J_{\mathrm{SK}}\cdot\nu)^+=J^+_{\mathrm{Hit}}\cdot\nu^+$. Hence we see that the spinor components of (4.20) and (4.21) do indeed reproduce the corresponding equations in (4.14).

In summary, we see that we have reproduced the $N=1$ vacuum equations in $E_{7(7)}$ form. The L equation is equivalent to the first equation in (4.14). Decomposing into $\mathrm{SL}(2,\mathbb{R})\times O(6,6)$ representations we see that spinor components of the K_a equations (4.20) and (4.21) are equivalent the the second and third equations in (4.14). However, it is harder to see how the vector components of these equations can be implied by supersymmetry. It seems likely that one would need to take further projections to get a consistent $E_{7(7)}$ form of the equations.

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5 Conclusions

In this paper we studied $N = 2$ backgrounds of type II string theory from a geometric viewpoint where the U-duality group is manifest. The formalism we have used, known as extended or exceptional generalised geometry, has been developed in refs. [27, 28] and is an extension of Hitchin's generalised geometry [7]. The latter framework "unifies" the tangent and cotangent bundles, such that the T-duality group acting on the degrees of freedom in the NS-sector is manifest. Incorporating the RR-sector requires a formalism where the full U-duality group acts on some even larger generalised tangent bundle.

For the case at hand the T-duality group is $O(6,6)$ and the NS degrees of freedom are most conveniently represented by two pure spinors $\Phi^{\pm}$. Each of them is invariant under an $\mathrm{SU}(3,3)$ action and thus parameterises a specific $O(6,6)$ orbit corresponding (after a quotient by $\mathbb{C}^*$) to the special Kähler cosets $\mathcal{M}_{\mathrm{SK}} = O(6,6)/\mathrm{U}(3,3)$. Together, given a compatibility condition, the two spinors define an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. The two $\mathrm{SU}(3)$ factors correspond to the invariance groups of the two six-dimensional spinors used to define the $N = 2$ background. Incorporating the RR degrees of freedom enlarges the T-duality group to the U-duality group $E_{7(7)}$ which contains $\mathrm{SL}(2,\mathbb{R}) \times O(6,6)$ as one of its subgroups. Furthermore, $N = 2$ requires that one of the special Kähler cosets is promoted to a quaternionic Kähler manifold by means of the c-map [20].

In this paper we showed that the NS and RR degrees of freedom which populate $N = 2$ hypermultiplets can be embedded into the **133** adjoint representation of $E_{7(7)}$ and that they parameterise an $E_{7(7)}$ orbit corresponding to the Wolf-space $\mathcal{M}_{\mathrm{QK}} = E_{7(7)}/(\mathrm{SO}^*(12) \times \mathrm{SU}(2))$ which is indeed quaternionic-Kähler. A point in the orbit is defined by a triplet of real elements K_a fixing an $\mathrm{SU}(2)$ subgroup of $E_{7(7)}$. The $\mathrm{SU}(2)$ action is the R-symmetry of the $N = 2$ theory. We also constructed the corresponding hyperkähler cone and hyperkähler potential following refs. [23, 34, 35] and showed agreement with the expressions given in [36, 37].

Similarly, we demonstrated that the degrees of freedom which reside in $N = 2$ vector multiplets can be embedded into a complex element L of the **56** fundamental representation of $E_{7(7)}$. This parameterises an $E_{7(7)}$ orbit, corresponding, after a $\mathbb{C}^*$ quotient, to the special Kähler coset $\mathcal{M}_{\mathrm{SK}} = E_{7(7)}/(E_{6(2)} \times \mathrm{U}(1))$. Again the $\mathrm{U}(1)$ factor is an R-symmetry, which combines with the $\mathrm{SU}(2)$ factor from the hypermultiplets to give $\mathrm{U}(2)_{\mathrm{R}}$. The corresponding Kähler potential is given by the logarithm of the square root of the $E_{7(7)}$ quartic invariant in complete analogy with the appearance of the Hitchin function invariant in the $O(6,6)$ formalism [7, 10, 11]. We noted that to fill out the full $E_{7(7)}$ orbit in the **56** representation requires generalising the original $N = 2$ spinor ansatz to its most generic form (3.57). These six-dimensional spinors transform in the fundamental representation of the local U-duality group $\mathrm{SU}(8)$. By considering an $N = 2$ background one picks out two $\mathrm{SU}(8)$ spinors defining an $\mathrm{SU}(6)$ structure which decomposes into $\mathrm{SU}(3) \times \mathrm{SU}(3)$ under the T-duality subgroup. Correspondingly, there is a compatibility condition between the hypermultiplet K_a and vector multiplet L such that together they are indeed invariant under a $\mathrm{SU}(6)$ subgroup of $E_{7(7)}$.

Finally, the Killing prepotentials (or moment maps) $\mathcal{P}_a$ which determine the scalar potential and couple the two sectors, can also be given in an $E_{7(7)}$ language. Unlike the kinetic term moduli spaces, the prepotentials depend on the differential structure of the exceptional generalised geometry. Nonetheless they also take a simple form in terms of $E_{7(7)}$ objects which we demonstrated agrees with the expressions calculated in [11]. We ended by considering supersymmetric $N = 1$ backgrounds. Without giving a complete description we showed that there are natural $E_{7(7)}$ expressions which encode the supersymmetry conditions. However, there are indications that these probably require additional projections to give a fully consistent $E_{7(7)}$ form.

There are a number of natural extensions of this work one could consider. First is of course to consider other dimensions and numbers of preserved supersymmetries [48]. The former corresponds to exceptional generalised geometry with different $E_{d(d)}$ U-duality groups, while the latter correspond to different preserved structures within these groups. It is important to note that the formulation presented here is not $E_{7(7)}$ covariant: in particular the $E_{7(7)}$ symmetry is broken by picking out the particular $GL(6,\mathbb{R})$ subgroup which acts on the tangent space. The actual symmetry group for the supergravity theory is a semidirect product of this diffeomorphism symmetry group with the gauge potential transformations given in (3.17). Together these form a parabolic subgroup of $E_{7(7)}$. Our formalism is covariant with respect to this subgroup. One could also consider more general so-called "non-geometrical" backgrounds [49], which in some sense incorporate more of the full U-duality group. From the evidence of the $O(6,6)$-formulation, one expects the $E_{7(7)}$ expressions for the kinetic and potential terms would hold in this more general context. Furthermore, there also seems to be an intriguing relation between the moduli spaces which we discussed in this paper and the charge orbits and moduli spaces of extremal black hole attractor geometries [50].[22]

One could also consider what this formulation implies for the topological string, which describes the vector multiplet sector of type II supergravity. It has been argued [51] that the $O(6,6)$ Hitchin functionals are the actions of the corresponding target space theories. In the $E_{7(7)}$ formalism, we have seen that the kinetic terms are now described by the analogue of a Hitchin function on $\mathbb{R}^+ \times E_{7(7)} \,/\, E_{6(2)}$ given by the square-root of the $E_{7(7)}$ quartic invariant. Interestingly, as in the $O(6,6)$ case this space is also a so-called "prehomogeneous space", implying it is an open orbit in the **56** dimensional representation. It would be very interesting to see how the one-loop calculations of ref. [52] are encoded in this U-duality context, and how this is connected to the extension of the topological string to M-theory.

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A A $GL(6,\mathbb{R})$ subgroup of $E_{7(7)}$

We would like to identify a particular embedding of $GL(6,\mathbb{R})$ in $\mathrm{SL}(2,\mathbb{R})\times O(6,6)\subset E_{7(7)}$. This will correspond to the action of diffeomorphisms on the exceptional tangent space in the EGG. For the $O(6,6)$ factor, we simply take the embedding of $GL(6,\mathbb{R})$ that arises in generalised geometry. If $a\in GL(6,\mathbb{R})$ acts on vectors $y\in TM$ as $y\mapsto ay$, then $GL(6,\mathbb{R})\subset O(6,6)$ acts on the fundamental **12** representation as, given $y+\xi\in TM\oplus T^*M$,

$$\begin{pmatrix} y \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^T \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} . \tag{A.1}$$

We also choose to embed $GL(6,\mathbb{R})$ in the $\mathrm{SL}(2,\mathbb{R})$ factor so that, an $\mathrm{SL}(2,\mathbb{R})$ doublet w^i transforms as

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \mapsto \begin{pmatrix} (\det a)^{-1/2} & 0 \\ 0 & (\det a)^{1/2} \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} . \tag{A.2}$$

Putting these two ingredients together implies that elements $\lambda=(\lambda^{iA},\lambda^+)$ of the **56** representation of $E_{7(7)}$, decomposing under $GL(6,\mathbb{R})$, transform as sections of a bundle

$$\begin{aligned} E_0 = \left(\Lambda^6 T^*M\right)^{-1/2} \otimes \Big[& TM \oplus T^*M \\ & \oplus \Lambda^5 T^*M \oplus \left(T^*M\otimes\Lambda^6T^*M\right) \oplus \Lambda^{\text{even}}T^*M \Big] , \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} \lambda^{1A} &\in \left(\Lambda^6T^*M\right)^{-1/2} \otimes \Big[\Lambda^5T^*M \oplus \left(T^*M\otimes\Lambda^6T^*M\right)\Big] , \\ \lambda^{2A} &\in \left(\Lambda^6T^*M\right)^{-1/2} \otimes \Big[TM\oplus T^*M\Big] , \\ \lambda^{+} &\in \left(\Lambda^6T^*M\right)^{-1/2} \otimes \Lambda^{\text{even}}T^*M \ . \end{aligned} \tag{A.4}$$

It will be helpful to also define spaces weighted by a power of Λ^6T^*M so

$$E_p = (\Lambda^6T^*M)^p \otimes E_0 , \tag{A.5}$$

such that

$$\begin{aligned} E \equiv E_{1/2} &= TM\oplus T^*M\oplus\Lambda^5T^*M\oplus\left(T^*M\otimes\Lambda^6T^*M\right)\oplus\Lambda^{\text{even}}T^*M , \\ E_{-1/2} &= TM\oplus T^*M\oplus\Lambda^5TM\oplus\left(TM\otimes\Lambda^6TM\right)\oplus\Lambda^{\text{even}}TM \ . \end{aligned} \tag{A.6}$$

Thus we can write a general element of E as

$$\lambda = y+\xi+\nu+\pi+\lambda^+ \in E , \tag{A.7}$$

where $y\in TM$, $\xi\in T^*M$, $\nu\in\Lambda^5T^*M$, $\pi\in T^*M\otimes\Lambda^6T^*M$ and $\lambda^+\in\Lambda^{\text{even}}T^*M$ such that the $(\mathbf{2},\mathbf{12})$ components are $\lambda^{2A}=y^m+\xi_m$ and $\lambda^{1m}=\nu^m_{1\dots6}$ (with $\nu^m_{m_1\dots m_6}=6\delta^m_{[m_1}\nu_{m_2\dots m_6]}$) and $\lambda^1_m=\pi_{m,1\dots6}$.

One can also make a corresponding decomposition of the adjoint representation. We find

$$\begin{aligned} A_0 = (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M \\ \oplus \mathbb{R} \oplus \Lambda^6 T^*M \oplus \Lambda^6 TM \oplus \Lambda^{\text{odd}} T^*M \oplus \Lambda^{\text{odd}} TM\,, \end{aligned} \tag{A.8}$$

where $\mu = (\mu^i{}_j, \mu^A{}_B, \mu^{i-}) \in A_0$ has

$$\begin{aligned} &\mu^1{}_1 = -\mu^2{}_2 \in \mathbb{R}\,, \qquad \mu^1{}_2 \in \Lambda^6 T^*M\,, \qquad \mu^2{}_1 \in \Lambda^6 TM\,, \\ &\mu^A{}_B \in (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M\,, \\ &\mu^{1-} \in \Lambda^{\text{odd}} T^*M\,, \qquad\qquad \mu^{2-} \in \Lambda^{\text{odd}} TM\;. \end{aligned} \tag{A.9}$$

We also define $A_p = (\Lambda^6 T^*M)^p \otimes A_0$.

Note that we can identify a subgroup of $E_{7(7)}$ generated by the forms in A_0. Introducing a vector v^i with $v^1 = 1$ and $v^2 = 0$, we can write them in a more covariant way as

$$\begin{aligned} \mu^i{}_j &= \tilde{B}_{1\dots 6} v^i v_j\,, & \tilde{B} &\in \Lambda^6 T^*M\,, \\ \mu^A{}_B &= \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, & B &\in \Lambda^2 T^*M\,, \\ \mu^{i-} &= v^i C^-\,, & C^- &\in \Lambda^{\text{odd}} T^*M\,, \end{aligned} \tag{A.10}$$

with the sub-algebra in $\mathfrak{e}_{7(7)}$

$$\left[B + \tilde{B} + C^-, B' + \tilde{B}' + C^{-\prime}\right] = 2\left\langle C^-, C^{-\prime}\right\rangle + B \wedge C^{-\prime} - B' \wedge C^-\,, \tag{A.11}$$

that is, the commutator corresponds to a transformation with $\tilde{B}'' = 2\langle C^-, C^{-\prime}\rangle$ and $C^{-\prime\prime} = B \wedge C^{-\prime} - B' \wedge C^-$. Note that this Lie algebra is nilpotent with index four. The adjoint action of the subalgebra on an element $\lambda = y + \xi + \nu + \pi + \lambda^+$ is given by

$$\begin{aligned} \left(B + \tilde{B} + C^-\right) \cdot \lambda = -i_y B + \left(i_y \tilde{B} + \left\langle C^-, \hat{\jmath}\lambda^+\right\rangle\right) \\ + \left(jB \wedge \nu + j\xi \wedge \tilde{B} + \left\langle C^-, j\lambda^+\right\rangle\right) + B \wedge \lambda^+\,, \end{aligned} \tag{A.12}$$

where we are using the notation that the symbol j denotes the pure T^*M index of $T^*M \otimes \Lambda^6 T^*M$ and the symbol $\hat{\jmath}$ denotes the TM index of $TM \otimes \Lambda^6 T^*M \simeq \Lambda^5 T^*M$. In particular, given for any one-form γ, the element $\langle C^-, \hat{\jmath}\lambda^+\rangle \in \Lambda^5 T^*M$ is given by

$$\gamma \wedge \left\langle C^-, \hat{\jmath}\lambda^+\right\rangle = \left\langle C^-, \gamma \wedge \lambda^+\right\rangle\;. \tag{A.13}$$

while the elements $jB \wedge \nu$ and $\langle C^-, j\lambda^+\rangle$ in $T^*M \otimes \Lambda^6 T^*M$ are given by, for any vector y^m,

$$\begin{aligned} y^m \left(jB \wedge \nu\right)_{m,m_1\dots m_6} &= \left(i_y B \wedge \nu\right)_{m_1\dots m_6}\,, \\ y^m \left(j\xi \wedge \tilde{B}\right)_{m,m_1\dots m_6} &= \left(i_y \xi\right) \tilde{B}_{m_1\dots m_6}\,, \\ y^m \left\langle C^-, j\lambda^+\right\rangle_{m,m_1\dots m_6} &= \left\langle C^-, i_y \lambda^+\right\rangle_{m_1\dots m_6}\;. \end{aligned} \tag{A.14}$$

B Computing DK_a

We would like to calculate the derivative DK_a where $D \in \mathbf{56}$ is the embedding of the exterior derivative given by (4.2) and in the action of D on K_a we project onto the $\mathbf{56}$ representation.

It will be useful to introduce explicit indices for the components of the $\mathbf{56}$ and $\mathbf{133}$ representations. Viewed as elements of the larger symplectic group $Sp(56,\mathbb{R}) \supset E_{7(7)}$ we can write $D^{\mathcal{A}}$, and $K_a^{\mathcal{AB}} = K_a^{\mathcal{BA}}$, where $\mathcal{A},\mathcal{B} = 1,\dots,56$. One then has $(DK_a)^{\mathcal{C}} = \mathcal{S}_{\mathcal{AB}} D^{\mathcal{A}} K_a^{\mathcal{BC}}$ where $\mathcal{S}_{\mathcal{AB}}$ is the symplectic structure (3.4). Given $\mu^{\mathcal{AB}} \in \mathbf{133}$ and some $E_{7(7)}$ group element g such that $\mu' = g\mu$ we have

$$
\begin{aligned}
(D\mu')^{\mathcal{C}} &= \mathcal{S}_{\mathcal{AB}} D^{\mathcal{A}} \left(g^{\mathcal{B}}{}_{\mathcal{B}'} g^{\mathcal{C}}{}_{\mathcal{C}'} \mu^{\mathcal{B}'\mathcal{C}'} \right) \\
&= g^{\mathcal{C}}{}_{\mathcal{C}'}\, g^{-1\mathcal{A}'}{}_{\mathcal{A}} \mathcal{S}_{\mathcal{A}'\mathcal{B}'} \left[D^{\mathcal{A}} \mu^{\mathcal{B}'\mathcal{C}'} + \mathcal{A}^{\mathcal{AB}'}{}_{\mathcal{B}} \mu^{\mathcal{BC}'} + \mathcal{A}^{\mathcal{AC}'}{}_{\mathcal{B}} \mu^{\mathcal{B}'\mathcal{B}} \right] ,
\end{aligned} \tag{B.1}
$$

where we have used $\mathcal{S}_{\mathcal{A}'\mathcal{B}'} g^{\mathcal{A}'}{}_{\mathcal{A}} g^{\mathcal{B}'}{}_{\mathcal{B}} = \mathcal{S}_{\mathcal{AB}}$ and have introduced the generalised connection

$$
\mathcal{A}^{\mathcal{AB}}{}_{\mathcal{C}} = g^{-1\mathcal{B}}{}_{\mathcal{D}} \left(D^{\mathcal{A}} g^{\mathcal{D}}{}_{\mathcal{C}} \right) \in \mathbf{56} \times \mathbf{133} \ . \tag{B.2}
$$

We now specialize to the case where $g = \mathrm{e}^{C-}$. Given $D = (v^i \mathrm{d}^A, 0)$ and using (3.13) and (3.6) we then have $g^{-1\mathcal{A}'}{}_{\mathcal{A}} D^{\mathcal{A}} = D^{\mathcal{A}'}$.[23] Hence the connection is given by

$$
\left(\mathrm{e}^{-C^-} D^{\mathcal{A}} \mathrm{e}^{C^-} \right)^{\mathcal{B}}{}_{\mathcal{C}} = D^{\mathcal{A}} \delta^{\mathcal{B}}{}_{\mathcal{C}} + \mathcal{A}^{\mathcal{AB}}{}_{\mathcal{C}} \ . \tag{B.3}
$$

This can then be calculated using a variant of the Baker-Campbell-Hausdorff formula which, in this context, reads

$$
\mathrm{e}^{-C^-} \mathrm{d}^A \mathrm{e}^{C^-} = \mathrm{d}^A \cdot \mathbf{1} + \mathrm{d}^A C^- + \frac{1}{2!} \left[\mathrm{d}^A C^-, C^- \right] + \frac{1}{3!} [[\mathrm{d}^A C^-, C^-], C^-] + \dots \ . \tag{B.4}
$$

This series truncates at second order with the only non-vanishing component[24]

$$
[\mathrm{d}^A C^-, C^-]^i{}_j = 2 v^i v_j \langle \mathrm{d}^A C^-, C^- \rangle \ . \tag{B.5}
$$

Given $(\mathrm{e}^{-C^-} D^{\mathcal{A}} K_a)^{\mathcal{BC}} = \left[(\mathrm{e}^{-C^-} D^{\mathcal{A}} \mathrm{e}^{C^-}) \mathrm{e}^{-B} K_a^{(0)} \right]^{\mathcal{BC}}$ and using (3.25) and the adjoint action of the generalised connection we find the nonzero components

$$
\begin{aligned}
\left(\mathrm{e}^{-C^-} D^{iA} K_+ \right)^j{}_k &= v^i \langle \mathrm{d}^A C^-, \Phi^- \rangle \left(v^j u_k + u^j v_k \right) , \\
\left(\mathrm{e}^{-C^-} D^{iA} K_+ \right)^B{}_C &= v^i (uv) \langle \mathrm{d}^A C, \Gamma^B{}_C \Phi^- \rangle , \\
\left(\mathrm{e}^{-C^-} D^{iA} K_+ \right)^{j-} &= v^i \mathrm{d}^A \left(u^j \Phi^- \right) - v^i (uv) \langle \mathrm{d}^A C^-, C^- \rangle v^j \Phi^- ,
\end{aligned} \tag{B.6}
$$

[23] Note that more generally all the form field transformations leave D invariant, that is, $\mathrm{e}^{-C^-} D = \mathrm{e}^{-\tilde{B}} D = \mathrm{e}^{B} D = D$.

[24] We have $v^i v_i = \epsilon^{ij} v_i v_j = 0$.

with $(\mathrm{e}^{-C^-}D^{\mathcal{A}}K_-)^{\mathcal{BC}}$ given by complex conjugation and

$$\begin{aligned}
(\mathrm{e}^{-C^-}D^{iA}K_3)^j{}_k &= \frac{1}{4}v^i\mathrm{d}^A\left[\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle(u^j\bar u_k+\bar u^j u_k)\right]-\frac{1}{4}v^i\mathrm{i}\kappa^{-1}\langle\Phi^-,\bar\Phi^-\rangle\\
&\qquad\times\langle\mathrm{d}^AC^-,C^-\rangle\left[(uv)(v^j\bar u_k+\bar u^j v_k)+(\bar uv)(v^ju_k+u^jv_k)\right]\ ,\\
(\mathrm{e}^{-C^-}D^{iA}K_3)^B{}_C &= -\frac{1}{4}v^i\mathrm{d}^A\left[\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle(-\mathrm{i}u\bar u)\mathcal{J}^B{}_C\right]\ ,\\
(\mathrm{e}^{-C^-}D^{iA}K_3)^{j-} &= -\frac{1}{4}v^i\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle\\
&\qquad\times\Big[\left((\bar uv)u^j+(uv)\bar u^j\right)\mathrm{d}^AC^- - \frac{1}{4}v^j(u\bar u)\mathcal{J}_{BC}\Gamma^{BC}\mathrm{d}^AC^-\Big]\ . \qquad \text{(B.7)}
\end{aligned}$$

Using (B.1) and (3.5) to project on the **56** component we then have

$$\begin{aligned}
\mathrm{e}^{-C^-}\left(DK_+\right)^{iA} &= v^i(uv)\left(\langle\mathrm{d}^AC^-,\Phi^-\rangle+\langle\Phi^-,\Gamma^A{}_B\mathrm{d}^BC^-\rangle\right)\ ,\\
\mathrm{e}^{-C^-}\left(DK_+\right)^+ &= \mathrm{d}\left[(uv)\Phi^-\right]\ , \qquad \text{(B.8)}
\end{aligned}$$

with again the complex conjugate expressions for DK^- and

$$\begin{aligned}
\mathrm{e}^{-C^-}\left(DK_3\right)^{iA} &= \frac{1}{4}\mathrm{d}^A\left[\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle\left((\bar uv)u^i+(uv)\bar u^i\right)\right]\\
&\quad-\frac{1}{4}v^i\mathrm{d}^B\left[\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle(-\mathrm{i}u\bar u)\mathcal{J}^A{}_B\right]\\
&\quad-\frac{1}{2}\kappa^{-1}v^i(uv)(\bar uv)\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle\langle\mathrm{d}^AC^-,C^-\rangle\ ,\\
\mathrm{e}^{-C^-}\left(DK_3\right)^+ &= -\frac{1}{2}\kappa^{-1}\mathrm{i}\langle\Phi^-,\bar\Phi^-\rangle(uv)(\bar uv)\mathrm{d}C^-\ . \qquad \text{(B.9)}
\end{aligned}$$

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